VOLUME 29, NUMBER 3, SEPTEMBER 2012

ALGEBRAS, GROUPS & GEOMETRIES Volume 29, Number 3, September 2012

# *ALGEBRAS, GROUPS AND GEOMETRIES*

**FOUNDED IN 1984. SOME OF THE PAST EDITORS INCLUDE PROFESSORS: A. A. ANDERSEN, V. V. DEODHAR, J. R. FAULKNER, M. GOLDBERG, N. IWAHORI, M. KOECHER, M. MARCUS, K. M. MCCRIMMON, R. V. MOODY, R. H. OHEMKE, J. M. OSBORN, J. PATERA, H. RUND, T. A. SPRINGER, G. S. TSAGAS.**

**EDITORIAL BOARD**

M.R. ADHIKARI
A.K. ARINGAZIN
A. BAYOUMI
G. CARICATO
QING-MING CHENG
A. HARMANCY
L. P. HORWITZ
C. P. JOHNSON
S.L. KALLA
L. KEXI
N. KAMIYA
A. U. KLIMYK
J. A. LESLIE*
J. LOHMUS*
R. MIRON
M.R. MOLAEI
P. NOWOSAD
KAR-PING SHUM
S. SILVESTROV
H. M. SRIVASTAVA
G.T. TSAGAS*
* In memoriam

***FOUNDER***
**R.M. SANTILLI**

***EDITOR IN CHIEF***
**S.S. MOSKALIUK**

HADRONIC PRESS, INC.

# ALGEBRAS, GROUPS AND GEOMETRIES

VOLUME 29, NUMBER 3, SEPTEMBER 2012

**CATEGORY OF TREES IN REPRESENTATION THEORY OF QUANTUM ALGEBRAS,** 247
**N.M. Moskaliuk and S.S. Moskaliuk**
Bogoliubov Institute for Theoretical Physics of NAS Ukraine
14b Metrolohichna St., Kyiv, Ukraine

**FINSLER GEOMETRY IN THE PRESENCE OF ISOTOPIC FIELD CHARGES APPLIED FOR GRAVITY,** 275
**György Darvas**
Symmetrion and Institute for Research
Organization of the Hungarian Academy of Sciences
18 Nádor St., Budapest, H-1051 Hungary

**ON THOMPSON'S CONJECTURE FOR ALTERNATING GROUP $A_{16}$,** 303
**Jianxing Bi**
School of Mathematics, Liaoning University
Shenyang, 110036, P.R. China

**ALGEBRA-INVARIANT MODELS FOR NONLINEAR NONLOCAL MEDIA,** 309
**Danylenko V.A., Danevich T.B., Makarenko O.S.[2], Ukraine, Moskaliuk N.M.[3], Skurativskiy S.I. and Vladimirov V.A.**
Subbotin Institute of Geophysics of the National Academy of Sciences of Ukraine
Palladin Av. 32, Kyiv, Ukraine, UA-03680

ALGEBRAS GROUPS AND GEOMETRIES 29, 247 - 274 (2012) 

# CATEGORY OF TREES IN REPRESENTATION THEORY OF QUANTUM ALGEBRAS*

**N.M. Moskaliuk** and S.S. Moskaliuk**
Bogoliubov Institute for Theoretical Physics of NAS Ukraine
14b Metrolohichna St., Kyiv, Ukraine
mss@bitp.kiev.ua

Received May 6, 2013

## Abstract

New applications of categorical methods are connected with new additional structures on categories. One of such structures in representation theory of quantum algebras, the category of Kuznetsov-Smorodinsky-Vilenkin-Smirnov-(KSVS)-trees, is constructed, whose objects are finite rooted KSVS-trees and morphisms generated by the transition from a KSVS-tree to another one.

---

* Based on invited talks given at the 5th Petrov International Symposium on *High Energy Physics, Cosmology and Gravity*, Kyiv (Ukraine), April 29–June 15, 2012 and partially supported by the Project No. 1202.094-12 of the Central European Initiative Cooperation Fund.

** This research was also co-financed by the Austrian Academy of Sciences in the framework of the collaboration with the National Academy of Sciences of Ukraine "Modern Problems in Astroparticle Physics".

Copyright © 2012 by Hadronic Press Inc., Palm Harbor, FL 34682, U.S.A.

## 1. INTRODUCTION

Method of trees was invented to elucidate some problems of Lie group representations, and it was developed in 1960-1970-th by Kuznetsov–Smorodinsky–Vilenkin (KSV) [1–5]. The further development of Kuznetsov-Smorodinsky-Vilenkin method to Kuznetsov–Smorodinsky–Vilenkin–Smirnov-(KSVS)-trees was given by Yu. F. Smirnov [6, 7] on the base of conception of complementarity of Lie groups (in sense of Moshinsky [8]) which exists between groups $O(n)$ and $Sp(2, R)$. In particular, problem of calculating of operator matrix elements in the method of $K$-harmonics was solved, which, in the long run, determined the practical significance of these results.

DEFINITION 1.1 *[9–12] A category is a quadruple* $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *consisting of:*

*(Cl) a class* Ob *of objects;*

*(C2) for each ordered pair* $(A, B)$ *of objects a set* $\mathrm{Hom}(A, B)$ *of morphisms;*

*(C3) for each object* $A$ *a morphism* $\mathrm{id}_A \in \mathrm{Hom}(A, A)$*, the identity of* $A$*;*

*(C4) a composition law associating to each pair of morphisms* $f \in \mathrm{Hom}(A, B)$ *and* $g \in \mathrm{Hom}(B, C)$ *a morphism* $g \circ f \in \mathrm{Hom}(A, C)$*;*

*which is such that:*

*(Ml)* $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* $f \in \mathrm{Hom}(A, B)$*,* $g \in \mathrm{Hom}(B, C)$ *and* $h \in \mathrm{Hom}(C, D)$*;*

*(M2)* $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ *for all* $f \in Hom(A, B)$*;*

*(M3) the sets* $\mathrm{Hom}(A, B)$ *are pairwise disjoint.*

Denote by $\mathtt{Tree}_{KSVS}$ as the category whose objects are finite rooted KSVS-trees with the following properties: a) the multiplicity of each vertex is at least two; b) at each vertex either all incoming flags are halves of edges, or all incoming flags are tails. Morphisms are generated by the following two classes of maps:

a) Isomorphisms compatible with orientation.

b) Contraction of all edges having a common vertex with some outgoing flag and keeping orientation.

More formally, a morphism $\varphi : \sigma \to \tau$ consists of two maps $\varphi_V : V_\sigma \to V_\tau$ and $\varphi^F : F_\tau \to F_\sigma$ compatible with boundaries and involutions and such that $\varphi^F$ sends tails to tails.

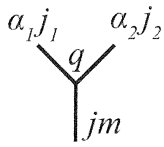
## 2. CATEGORY OF KSVS-TREES IN REPRESENTATION THEORY OF QUANTUM ALGEBRA $SU_Q(2)$

For basis vectors $|jm\rangle$ of irreducible representations $D^j$ of quantum algebra $su_q(2)$ and their products we use the same notations (lines and branchings) as in the usual algebra $su(2)$ [1]. For example, vector addition of two momenta $j_1$ and $j_2$ in the case of quantum algebra $su_q(2)$, like in the usual quantum theory of angular momentum, can be carried out, using Clebsch–Gordan $q$-coefficients:

$$|j_1 j_2 : jm\rangle_q = \sum_{m_1 m_2} (j_1 m_1, j_2 m_2 \mid jm)_q \mid j_1 m_1\rangle \mid j_2 m_2\rangle \ . \qquad (1)$$

But at the graph of vector sum of the momenta $j_1$ and $j_2$ it is necessary to indicate the value of the fixed parameter $q$, which characterizes the given

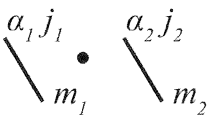
quantum algebra $su_q(2)$, because it is this value on which the value of Clebsch–Gordan coefficients $(j_1 m_1, j_2 m_2 \,|\, jm)_q$, using which the "vector addition of momenta" takes place (or the form of co-product — to speak in the terms of Hopf algebra). Thus, the graphic interpretation of the relation (1) is as follows [2]:




$$= |\alpha_1 j_1, \alpha_2 j_2 : jm\rangle_q \quad =$$

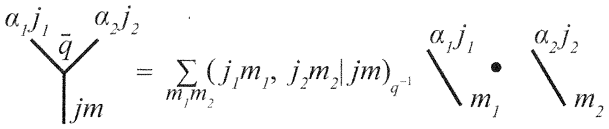
$$= \sum_{m_1 m_2} (j_1 m_1,\ j_2 m_2 | jm)_q \,|\, \alpha_1 j_1 m_1\rangle \ |\alpha_2 j_2 m_2\rangle =$$

$$=$$




$$\sum_{m_1 m_2} (j_1 m_1,\ j_2 m_2 | jm)_q \tag{2}$$

Along with such branching the branching of the type



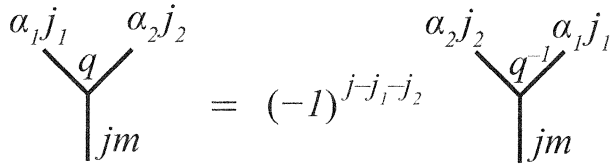

(3)

are required as well. This is related to the fact that the permutation property of symmetry for $q$-analogue of Clebsch–Gordan coefficients are

$$(j_1 m_1,\ j_2 m_2 \,|\, jm)_q = (-1)^{j - j_1 - j_2} (j_2 m_2,\ j_1 m_1 \,|\, jm)_{q^{-1}} \,. \tag{4}$$

For this reason the permutation of branches in the branching of KSVS-tree leads not only to arising of a phase factor, but also to the substitution of $q$

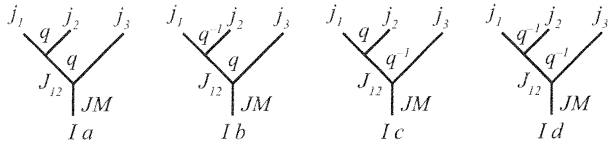
for $q^{-1}$:




(5)

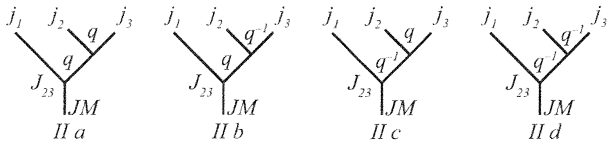
and at complex tree-like graphs may present the vertices with index $q$ as well as index $q^{-1}$.

In the case of addition of three momenta instead of one KSVS-tree, corresponding to the coupling scheme $((\mathbf{j}_1+\mathbf{j}_2)+\mathbf{j}_3)$ for the usual algebra $su(2)$, there are four types of KSVS-trees, because to each vertex we can ascribe parameter either $q$ or $q^{-1}$:



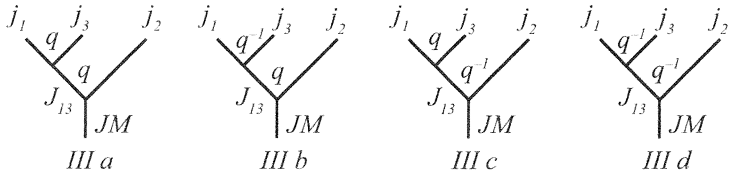

(6)

To the coupling scheme $(\mathbf{j}_1+(\mathbf{j}_2+\mathbf{j}_3))$ there correspond another four KSVS-trees:




(7)

One has to add here four KSVS-trees more with another pair of the branches bound in the upper branching:




(8)

KSVS-trees which can be turned into each other by permutation of branches in the branching are called equivalent. Any KSVS-tree with three dangling (free) branches $j_1, j_2, j_3$ and arbitrary ascribing of parameters $q$ and $q^{-1}$ to the vertices is equivalent to one of the KSVS-trees at (6)–(8).

Now, we shall be interested in morphisms generated by the transition from one KSVS-tree to another. It is worth of mentioning that the transitions between KSVS-trees $Ia$, $Ib$, $Ic$, $Id$ as well as between KSVS-trees $IIa$, $IIb$, $IIc$, $IId$ and $IIIa$, $IIIb$, $IIIc$, $IIId$ are impossible due to the fact that the KSVS-trees $a$, $b$, $c$, $d$, each in its own set, correspond to different variants of the choice of operator of total momentum $J_{\pm}(123)$, i.e., to different ways of co-multiplication in the spaces of three irreducible representations $D^{j_1} \otimes D^{j_2} \otimes D^{j_3}$. For example, for the trees at (6) one can write that

$$\begin{aligned} Ia) \qquad & J^{qq}_{\pm}(12,3) = J_{\pm}(1)\, q^{J_0(2)+J_0(3)} + J_{\pm}(2)\, q^{-J_0(1)+J_0(3)} + \\ & + J_{\pm}(3)\, q^{-J_0(1)-J_0(2)} = J^{q}_{\pm}(12)\, q^{J_0(3)} + J_{\pm}(3)\, q^{-J_0(12)} = \\ & = J_{\pm}(1)\, q^{J_0(23)} + J^{q}_{\pm}(23)\, q^{-J_0(1)} = J^{qq}_{\pm}(1,23)\,, \quad \text{i.e. } Ia \sim IIa\,; \end{aligned} \tag{9}$$

$$\begin{aligned} Ib) \qquad & J^{q^{-1}q}_{\pm}(12,3) = J_{\pm}(1)\, q^{-J_0(2)+J_0(3)} + J_{\pm}(2)\, q^{J_0(1)+J_0(3)} + \\ & + J_{\pm}(3)\, q^{-J_0(1)-J_0(2)} = J^{q^{-1}}_{\pm}(12)\, q^{J_0(3)} + J_{\pm}(3)\, q^{-J_2(1)-J_0(2)} = \\ & = J^{q}_{\pm}(13)\, q^{-J_0(2)} + J_{\pm}(2)\, q^{J_0(1)+J_0(3)} = J^{qq}_{\pm}(13,2)\,, \text{ i.e. } Ib \sim IIIc\,; \end{aligned} \tag{10}$$

$$\begin{aligned} Ic) \qquad & J^{qq^{-1}}_{\pm}(12,3) = J_{\pm}(1)\, q^{J_0(2)-J_0(3)} + J_{\pm}(2)\, q^{-J_0(1)-J_0(3)} + \\ & + J_{\pm}(3)\, q^{J_0(1)+J_0(2)} = J^{q}_{\pm}(12)\, q^{-J_0(3)} + J_{\pm}(3)\, q^{J_0(1)+J_0(2)} = \\ & = J^{q^{-1}}_{\pm}(13)\, q^{J_0(2)} + J_{\pm}(2)\, q^{-J_0(1)-J_0(3)} = J^{q^{-1}q}_{\pm}(13,2)\,, \text{ i.e. } Ic \sim IIIb\,; \end{aligned} \tag{11}$$

$$Id) \qquad J_{\pm}^{q^{-1}q^{-1}}(12,3) = J_{\pm}(1)\, q^{-J_0(2)-J_0(3)} + J_{\pm}(2)\, q^{J_0(1)-J_0(3)} + \tag{12}$$

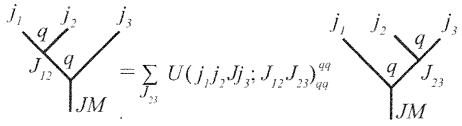
$$+J_{\pm}(3)\, q^{J_0(1)+J_0(2)} = J_{\pm}^{q^{-1}}(12)\, q^{-J_0(3)} + J_{\pm}(3)\, q^{J_0(1)+J_0(2)} =$$

$$= J_{\pm}(1)\, q^{-J_0(2)-J_0(3)} + J_{\pm}^{q^{-1}}(23)\, q^{J_0(1)} = J_{\pm}^{q^{-1}q^{-1}}(1,23)\,, \text{ i.e. } Id \sim IId\,.$$

As for operator $J_0(123)$, the following relation $J_0(123) = J_0(1)+J_0(2)+J_0(3)$ is valid for all trees $I$–$III$. It can be seen from (9) that for diagrams $Ia$ and $IIa$ the total momentum $J_{\pm}(123)$ is determined in the same way, and for this reason the transition between aforementioned trees by the replanting of the branch $j_2$ is possible and can be described as

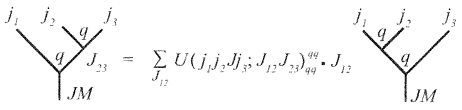
$$|j_1 j_2(J_{12}),\, j_3 : JM\rangle_{qq} = \sum_{J_{23}} U(j_1 j_2 J j_3;\; J_{12} J_{23})_{qq}^{qq} |j_1, j_2 j_3(J_{23});\; JM\rangle_{qq} \tag{13}$$

or graphically




(14)

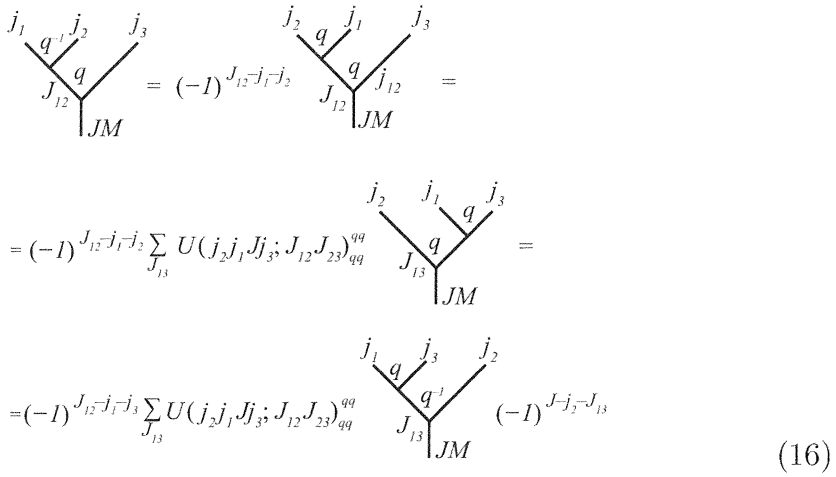
The morphism generated by the opposite transition can be carried out using the same coefficients




(15)

Such replantation we call standard. The comparison with the expressions for operator of "total momentum" $J_{\pm}(123)$ of another trees shows that the direct transition from the KSVS-tree $Ia$ to another KSVS-trees $I, III$ (except for $IIa$) is impossible without so-called universal $R$-matrix (see next Section).

It follows from (11) that the direct transition is possible only between the KSVS-trees $Ib$ and $IIIc$. In this case



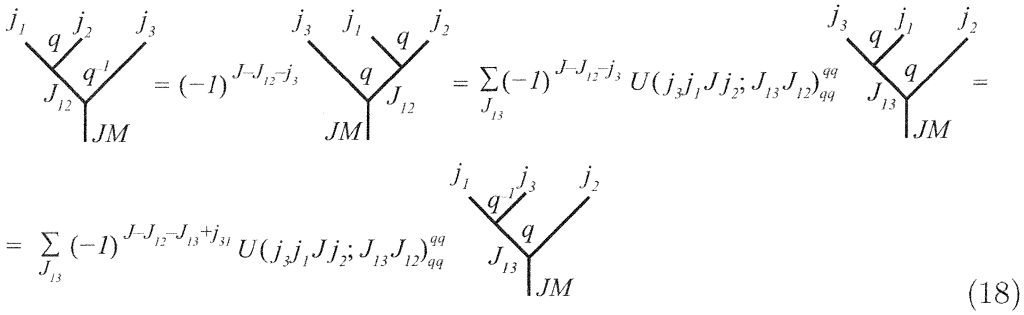

(16)

or

$$|j_1 j_2(J_{12}),\ j_3 : JM\rangle_{q\bar{q}} = \sum_{J_{13}} (-1)^{J-J_{13}-J_{12}+j_1} \times$$
$$\times U\left(j_2 j_1 J j_3;\ J_{12} J_{23}\right)^{qq}_{qq} |\, j_1 j_3(J_{13}),\ j_2 : JM\rangle_{qq^{-1}}\,. \qquad (17)$$

From the calculations brought about above it is clear that the non-standard transition from the KSVS-tree $Ib$ to the KSVS-tree $IIIc$ can be carried out using the standard Racah coefficients and phase factors. The relation (12) shows that the direct transition from the KSVS-tree $Ic$ is possible only to

the KSVS-tree $IIIb$ or the KSVS-tree equivalent to the latter:

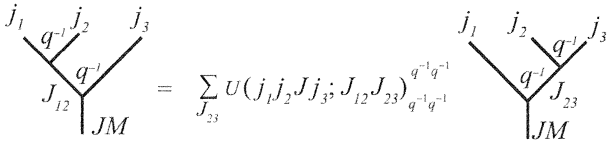






(18)

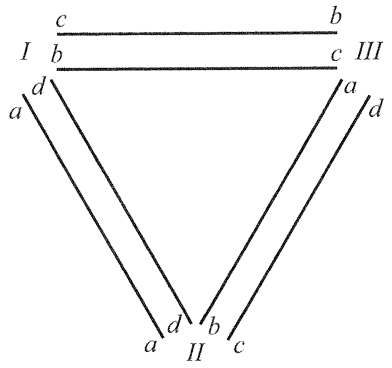
The possibility of the unique direct transition follows from (13):




(19)

This transition is similar to transition (14), but at another choice of $q \to q^{-1}$, and it can be omitted. But it follows from the general formula for Racah coefficients that they do not depend on $q$. For this reason in (9) enter really the same standard Racah coefficients as (14). It follows from relations (14)–(19) that the Racah coefficients with other sets of the right indices, except for $U(\dots)^{qq}_{qq}$, do not occur in the problem on the direct transitions between the KSVS-trees with three dangling branches $j_1, j_2, j_3$. Such transitions are called transitions of the first kind, and in the notations of the standard Racah coefficients $U(\dots)^{qq}_{qq}$ we shall indicate only one index $q$. The possible morphisms generated by transitions of the first kind between the KSVS-

trees $I$–$III$ (or equivalent to them) are shown as follows:



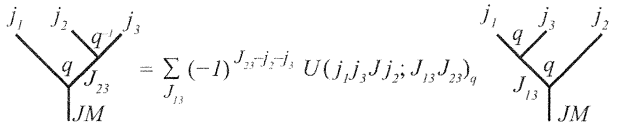

(20)

It can be seen that it is necessary to supplement the relations (9)–(13) by two another relations:

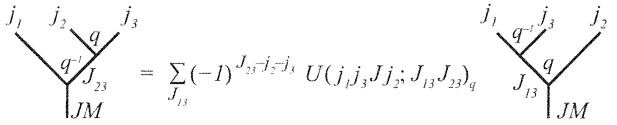
$$IIb) \quad J_+^{q\bar{q}}(1,23) = J_\pm(1)\, q^{J_0(2)+J_0(3)} + J_+(2)\, q^{-J_0(1)-J_0(3)} +$$
$$+J_+(3)\, q^{-J_0(1)+J_0(2)} = J_+(1)\, q^{J_0(2)+J_0(3)} + J_0^{q^{-1}}(23)\, q^{-J_0(1)} =$$
$$= J_+^{q}(13)\, q^{J_0(2)} + J_+(2)\, q^{-J_0(1)-J_0(3)} = J_0^{qq}\,(13,2)\,,$$
$$\text{i.e.} \quad IIb \sim IIIa\,; \tag{21}$$

$$IIc) \quad J_+^{\bar{q}q}(1,23) = J_+(1)\, q^{-J_0(2)-J_0(3)} + J_+(2)\, q^{J_0(1)+J_0(3)} +$$
$$+J_+(3)\, q^{J_0(1)-J_0(2)} = J_+(1)\, q^{-J_0(23)} + J_+^{q}(23)\, q^{J_0(1)} \times$$
$$\times J_+^{q^{-1}}(13)\, q^{-J_0(2)} + J_\pm(2)\, q^{J_0(1)+J_0(3)} = J_\pm^{q^{-1}q^{-1}}\,(13,2)\,,$$
$$\text{i.e.} \quad IIc \sim IIId\,. \tag{22}$$

In fact these transformations are carried out as follows:



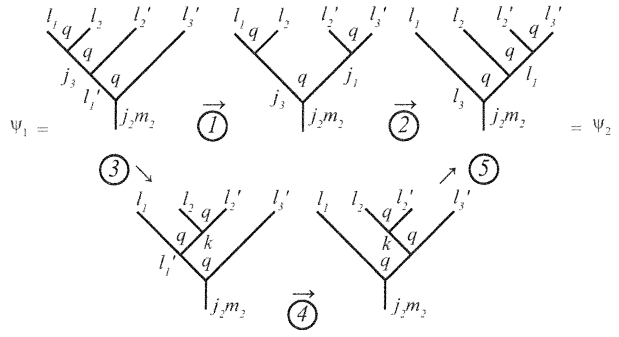

(23)




(24)

It can be seen from relations (14)–(24) that the transitions of the first kind between any two trees can be carried out only in the case when they can be reduced to the permutation of branches and a series of standard replantations, i.e., such replantations, when the *same values of parameter* $qq$ *and* $\bar{q}\bar{q}$ are ascribed to both vertices, affected by replantation, *in both trees.* This requirement restricts the possibility of transition of the first kind between the trees and is peculiar for the method of trees for quantum algebra $su_q(2)$ making it different from the usual quantum theory of angular momentum, where the direct transition is possible from any tree to any other one. As an example of transition of the first kind between the trees with four dangling branches we produce graphic proof of $q$-analogue of Elliott–Biedenharn identity. Let us consider the transition from vectors $\psi_1 = |\,(l_1 l_2\,(j_3),\,(l_2')\,(l_1');\,l_3' \,:\, j_2 m_2\rangle_{qqq}$ to vectors $\psi_2 = |\,l_1;\,(l_2,\,l_2' l_3'\,(j_1))\,(l_3) \,:\, j_2 m_2\rangle_{qqq}$, which can be carried out along two

routes:




(25)

The first route via the standard replantations *1* and *2* gives for the expansion coefficients $\langle \psi_1 \,|\, \psi_2 \rangle$ the following expression

$$\langle \psi_1 \,|\, \psi_2 \rangle = U\,(j_3 l_2' j_2 l_3';\; l_1' j_1)_q \, U\,(l_1 l_2 j_2 j_1;\; j_3 l_3)_q \,. \tag{26}$$

The second route — via the replantations *3–5* — leads to the result involving an additional sum over intermediate momentum $k$:

$$\langle \psi_1 \,|\, \psi_2 \rangle = \sum_k U(l_1 l_2 l_1' l_2';\; j_3 k)_q U(l_1 k j_2 l_3';\; l_1' l_3)_q U(l_2 l_2' l_3 l_3';\; k j_1)_q \,. \tag{27}$$

Equating the expressions (26) and (27) and making the transition from Racah coefficients to $6j$-symbols according to the formula [2]

$$U\,(abcd;\; ef)_q = (-1)^{a+b+c+d} \sqrt{[2e+1]\,[2f+1]} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q \,, \tag{28}$$

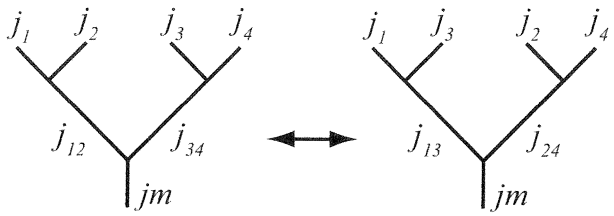
we come to the relation

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}_q \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1' & l_2' & l_3' \end{Bmatrix}_q = \sum_k (-1)^{k+j_1+j_2+j_3+l_1+l_2+l_3+l_1'+l_2'+l_3'} \times$$

$$\times [2k+1] \begin{Bmatrix} l_1 & k & l' \\ l_3' & j_2 & l_3 \end{Bmatrix}_q \begin{Bmatrix} l_2 & k & l_2' \\ l_1' & j_3 & l_1 \end{Bmatrix}_q \begin{Bmatrix} l_3 & k & l_3' \\ l_2' & j_1 & l_2 \end{Bmatrix}_q \,, \tag{29}$$

which is $q$-analogue of Elliott–Biedernharn identity for $6j$-symbols. To transform (26) and (27) into (29), it is necessary to use the symmetry properties of $q$-analogues of $6j$-symbols, which in essence are the same as in the case of usual $6j$-symbols (see [1]) .

When the transition under consideration does not belong to the first kind, at such transition from one KSVS-tree to another the structure of operators of the total momentum $J_{\pm}(1, 2, \ldots, n)$ must be changed, i.e., the choice of the scheme for co-multiplication in the space $D^{j_1} \otimes D^{j_2} \otimes \cdots \otimes D^{j_n}$, which can be realized using the universal $R$-matrix, has to be changed. The transitions of this type between the KSVS-trees can be called transitions of the second kind [2]. Matrices of such transitions, as it will be shown below, are elements of the corresponding $R$-matrices and can be reduced to phase factors, standard Racah coefficients and some powers of factors $q$. The necessity to use $R$-matrix we meet at the attempt to construct the $q$-analogue of $9j$-symbol. In fact, in quantum theory of angular momentum $9j$-symbol usually is introduced via matrix of the transition between the trees




(30)

However, in the case of algebra $su_q(2)$ (if to all branchings of both trees the parameter $q$ is ascribed) the straight-forward transition between these trees is impossible. For this reason the "total momentum" $j$ for the left tree

corresponds to operators

$$J_{\pm}^{qqq}(12,\,34) = J_{\pm}(1)q^{J_0(2)+J_0(3)+J_0(4)} + J_{\pm}(2)q^{-J_0(1)+J_0(3)+J_0(4)} + \\ + J_{\pm}(3)q^{-J_0(1)-J_0(2)+J_0(4)} + J_{\pm}(4)q^{-J_0(1)-J_0(2)-J_0(3)}\,, \quad (31)$$

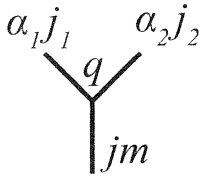
and for the right tree — to another operators

$$J_{\pm}^{qqq}(13,\,24) = J_{\pm}(1)q^{J_0(2)+J_0(3)+J_0(4)} + J_{\pm}(2)q^{-J_0(1)-J_0(3)+J_0(4)} + \\ + J_{\pm}(3)q^{-J_0(1)-J_0(2)+J_0(4)} + J_{\pm}(4)q^{-J_0(1)-J_0(2)-J_0(3)}\,. \quad (32)$$

It is obvious that $J_{\pm}^{qqq}(12,\,34) \neq J_{\pm}^{qqq}(13,\,24)$, i.e., the "total momentum" has different sense for different schemes of multiplication of representations. This fact we have called "non-conservation of total momentum" under the transition (30). In this connection, there is no sense in expanding of the vector, corresponding to the left tree at (30), in terms of vectors corresponding to the right tree, because such expansion besides sum over $j_{13}$, $j_{24}$ contains the sum over $j'$ as well. The latter is connected with the fact that due to the "non-conservation of total momentum" under the transition (30), the scalar product of vectors $|\,j_1j_2(j_{12}),\,j_3j_4(j_{34}) : jm\rangle_{qqq}$ and $|\,j_1j_3(j_{13}),\,j_2j_4(j_{24}) : j'm\rangle_{qqq}$ does not vanish for $j \neq j'$. Nobody has succeeded in "conservation of total momentum" $j$ for any other ways of independent ascribing parameters $q$ and $q^{-1}$ to branchings of both trees at (30). To assure the "conservation of momentum", as it takes place in the usual quantum theory of angular momentum, one can, only applying to any of the trees at (30) the universal $R$-matrix. To its graphic interpretation we now proceed.

## 3. GRAPHIC IMAGE OF $R$-MATRIX

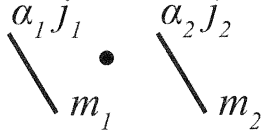
The action of operator $R_q^{j_1 j_2}$ on the vector $|\, j_1 j_2 : jm\rangle_q$ we depict graphically as intertwining of branches $j_1$, $j_2$ of the corresponding tree [2]




$$= |\alpha_1 j_1, \alpha_2 j_2 : jm\rangle_q \quad =$$

$$= \sum_{m_1 m_2} (j_1 m_1,\ j_2 m_2 | jm)_q \,|\, \alpha_1 j_1 m_1\rangle \ |\alpha_2 j_2 m_2\rangle =$$

$$= \sum_{m_1 m_2} (j_1 m_1,\ j_2 m_2 | jm)_q$$

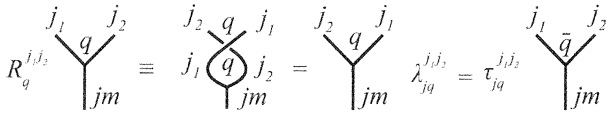



(33)

As a result, the equality

$$R_q^{12}\, J_\pm^q\ (12)\ (R_q^{12})^{-1} = J_\pm^q\ (21), \tag{34}$$

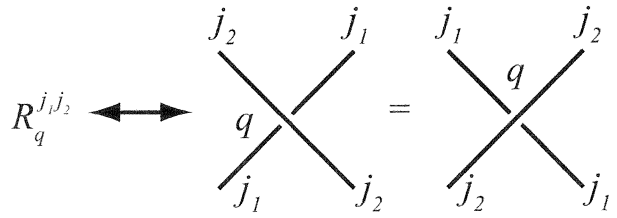
which is definition of $R$-matrix, can be presented by the following graph:




(35)

Doing so, in fact we have put in correspondence to $R$-matrix the intersection

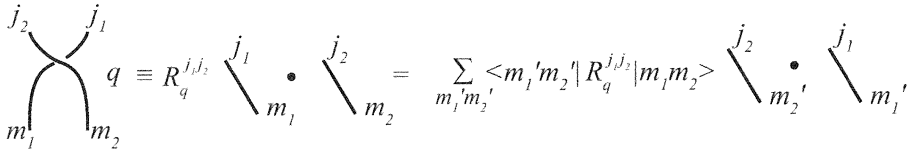
of two lines $j_2$ and $j_1$:




(36)

The disposition of the lines $j_1$ and $j_2$ in itself is unessential (for example, two "crosses" at (36) are turned in respect to each other by 90°, but this does not change $R$-matrix). The only thing, which is essential, is which of the lines is continuous and which is disrupted.

We shall keep to the following commonly accepted rule (irrespectively of the general disposition of lines) [2]: to the right upper index of $R$-matrix we put in correspondence continuous line, and to the left upper index — the interrupted line. The value of parameter $q$ ($q^{-1}$) is indicated near the point of intersection of these two lines. $R$-matrix enables to intertwine not only branches, combined in a united tree, but separate lines as well, for example[1]




(37)

[1] Graphic image of $R$-matrix should be situated above the graph of that very vector, on which it acts, i.e., the tree, involving $R$-matrix tangling, must be written analytically from above downward.

Algebraic formula of this equality is as follows

$$R_q^{j_1 j_2} |j_1 m_1\rangle \, |j_2 m_2\rangle = \sum_{m_1' m_2'} \langle m_1' m_2' \,|\, R_q^{j_1 j_2} \,|\, m_1 m_2\rangle \, |\, j_2 m_2'\rangle \, |\, j_1 m_1'\rangle \,. \qquad (38)$$

Thus, according to (35), $R$-matrix enables to transpose two branches in one branching of a tree *without substitution* $q \to q^{-1}$. Using the definition (35) and taking into account (38), we obtain

$$\sum_{m_1 m_2} \langle m_1' m_2' \,|\, R_q^{j_1 j_2} \,|\, m_1 m_2\rangle \, (j_1 m_1,\, j_2 m_2 \,|\, j m)_q = \\ = \lambda_{jq}^{j_1 j_2} \, (j_2 m_2',\, j_1 m_1' \,|\, j m)_q = \tau_{jq}^{j_1 j_2} \, \langle j_1 m_1' j_2 m_2' \,|\, j m\rangle_{\bar q} \qquad (39)$$

or, keeping in mind the orthonormality of Clebsch–Gordan $q$-coefficients,

$$\langle m_1' m_2' \,|\, R_q^{j_1 j_2} \,|\, m_1 m_2\rangle \equiv \langle j_1 m_1' j_2 m_2' \,|\, R_q^{12} \,|\, j_1 m_1 j_2 m_2\rangle = \\ = \sum_j \, (j_1 m_1',\, j_2 m_2' \,|\, j m)_{\bar q} \, \tau_q^{j_1 j_2} \, (j_1 m_1,\, j_2 m_2 \,|\, j m)_{\bar q} = \\ = \sum_j \, (j_2 m_2',\, j_1 m_1' \,|\, j m)_q \, \lambda_{jq}^{j_1 j_2} \, (j_1 m_1,\, j_2 m_2 \,|\, j m)_q \,. \qquad (40)$$

The latter means that $R$-matrix can be written as

$$R_q^{j_1 j_2} = \sum_{jm} \lambda_{jq}^{j_1 j_2} \, |\, j_2 j_1 : j m\rangle_q \, \langle j_1 j_2 : j m \,|_q \,. \qquad (41)$$

The explicit expression for $R$-matrix in terms of operators $J_\pm(1)$, $J_\pm(2)$ has been derived in [2]. It has been shown there as well that

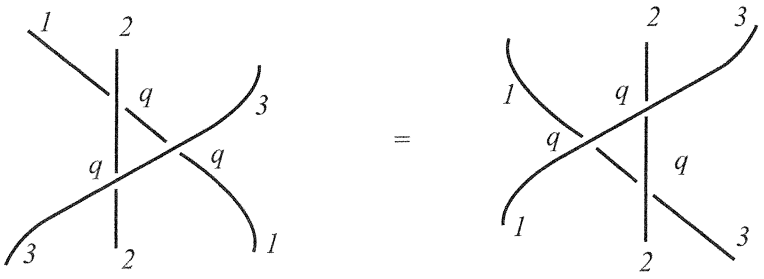
$$\lambda_{jq}^{j_1 j_2} = (-1)^{j - j_1 - j_2} \, q^{c(j) - c(j_1) - c(j_2)}, \quad c\,(j) = j\,(j+1) \,. \qquad (42)$$

It is worth of mentioning another useful relation. Acting on both sides of equality (35) by operator $(R_q^{j_1 j_2})^{-1}$ we acquire possibility to disintertwine the branches [2].

Above only one aspect of $R$-matrix has been of interest for us: the possibility of intertwining the branches of trees on its base. But it has another important property as well, satisfying Yang–Baxter equation

$$R_q^{12}\, R_q^{13}\, R_q^{23} = R_q^{23}\, R_q^{13}\, R_q^{12}\ . \tag{43}$$

Graphically Yang–Baxter equation is depicted as follows




(44)

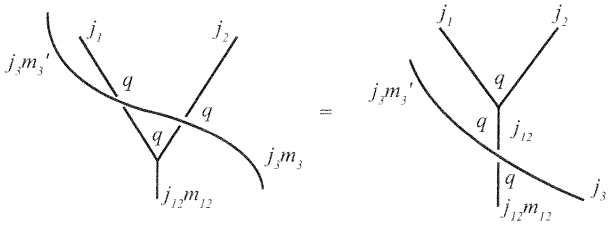
To prove it, we have used in [2] an auxiliary relation for $R$-matrix

$$R_q^{13}\, R_q^{23} = R_q^{12,3}\,. \tag{45}$$

Applying both sides of (45) to the vector $|j_1 j_2 : j_{12} m_{12}\rangle_q \,|\, j_3 m_3\rangle$, we come to identity

$$R_q^{j_1 j_3}\, R_q^{j_2 j_3}\,|\, j_1 j_2 : j_{12} m_{12}\rangle_q \,|\, j_3 m_3 = R_q^{12,3}\,|\, j_1 j_2 : j_{12} m_{12}\rangle_q \,|\, j_3 m_3\rangle\,, \tag{46}$$

with the following graphic image

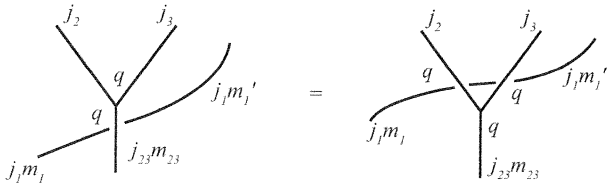



(47)

Turning to Hermitean adjoint of (45), one can derive another relation between $R$-matrices

$$R_q^{13}\, R_q^{12} = R_q^{1,23} \tag{48}$$

in the right side of which involves the operator $J_-^q(23)$. Applying both sides of (46) to the vector $|\, j_1 m_1\rangle\, |\, j_2 j_3 : j_{23} m_{23}\rangle_q$, we obtain the graphic identity




(49)

The equalities (47) and (49) mean essentially that if a line (continuous or interrupted) cuts both branches outgoing from some branching of a tree, that it can be drawn lower than the vertex, from which both branches go out. Therefore such line intersects only one line, namely the edge going in the aforementioned vertex. This possibility will be repeatedly used further.

## 4. MORPHISMS BETWEEN KSVS-TREES WITH INTERTWINED BRANCHES

As it has been mentioned in Section II, the morphisms generated by direct transition between the trees $Ia$ and $IIIa$ are impossible due to the non-conservation of total momentum under such transition. However, applying matrix $R_q^{32}$ to the momentum $J_\pm^{qq}(13, 2)$, this operator, according to the formula [2]

$$R_q^{12}\, J_\pm^q\ (12)\ (R_q^{12})^{-1} = J_\pm^q\ (21), \tag{50}$$

can be transformed into $J_{\pm}^{qq}(12,3)$:

$$R_q^{32} J_{\pm}^{qq}(13,\,2)\, R_{q^{-1}}^{32} = R_q^{32} \Big( J_{\pm}(1)\, q^{J_0(2)+J_0(3)} +$$
$$+ (J_{\pm}(2)\, q^{-J_0(3)} + J_{\pm}(3)\, q^{J_0(2)})\, q^{-J_0(1)} \Big)\, R_{q^{-1}}^{32} =$$
$$= J_{\pm}(1)\, q^{J_0(2)+J_0(3)} + (J_{\pm}(2)\, q^{+J_0(3)} +$$
$$J_{\pm}(3)\, q^{-J_0(2)})\, q^{-J_0(1)} = J_{\pm}^{qq}(12,\,3)\,.) \qquad (51)$$

As a result, the functions

$$\psi_{jm} = R_q^{32}\, |\, j_1 j_3\,(j_{13}),\; j_2 : jm\rangle_{qq} \varphi_{j'm'} = |\, j_1 j_2\,(j_{12}),\; j_3 : j'm'\rangle_{qq}$$

are eigenfunctions of Casimir operator

$$C_2^{12,3} = J_-^q(12,3)\, J_+^q(12,3) + [J_0\,(12,3) + 1/2]^2\,.$$

Therefore, they are mutually orthogonal if they belong to different eigenvalues of this operator

$$\langle \psi_{jm}\,|\,\varphi_{j'm'}\rangle = 0\,, \qquad j \neq j'\,, \quad m \neq m'\,. \qquad (52)$$

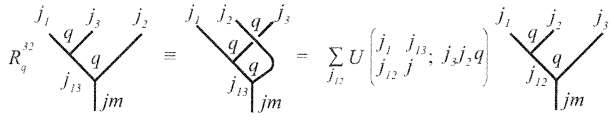
In this case vector $\psi_{jm}$ can be expanded over the complete set of orthonormal vectors $\varphi_{jm}$:

$$R_q^{32}\, |\, j_1 j_3(j_{13}),\; j_2 : jm\rangle_{qq} = \sum_{j_{12}} U \begin{pmatrix} j_1 & j_{13} & \\ & & ;\; j_3 j_2 q \\ j_{12} & j & \end{pmatrix} |j_1 j_2\,(j_{12}), j_3 : jm\rangle_{qq} \qquad (53)$$

The coefficients of expansion

$$U \begin{pmatrix} j_1 & j_{13} & \\ & & ;\; j_3 j_2 q \\ j_{12} & j & \end{pmatrix} \equiv \langle j_1 j_2\,(j_{12}), j_3 : jm\,|\, R_q^{32}\,|\, j_1 j_3(j_{13}),\; j_2 : jm\rangle_{qq}) \qquad (54)$$

are called in [2] Racah coefficients of the second kind. The graphic image of (53) is as follows




(55)

From (54) one can express Racah coefficients of the second kind in terms of Clebsch–Gordan $q$-coefficients

$$\sum_{\substack{m_1 m_2 m_2' \\ m_{12} m_3 \\ m_3' m_{13}}} (j_1 m_1,\, j_2 m_2 \,|\, j_{12} m_{12})_q \,(j_{12} m_{12},\, j_3 m_3 \,|\, j' m')_q \,(j_1 m_1,\, j_3 m_3 \,|\, j_{13} m_{13})_q$$

$$\times (j_{13} m_{13},\, j_2 m_2 \,|\, j m)_q \,\langle j_3 m_3' j_2 m_2' \,|\, R_q^{32} \,|\, j_3 m_3,\, j_2 m_2 \rangle$$

$$= \delta_{jj'} \delta_{mm'} U \begin{pmatrix} j_1 & j_{13} & \\ & & ;\; j_3 j_2 q \\ j_{12} & j & \end{pmatrix} . \tag{56}$$

It is important also to notice that vectors (53) with different values $j_{13} \neq j_{13}'$ are not mutually orthogonal and normed to the unit. Their scalar product one can evaluate, rebinding momenta

$$R_q^{32} \,|\, j_1 j_3 (j_{13}),\, j_2 : jm \rangle_{qq} =$$

$$= R_q^{32} \sum_{j_{23}} U\,(j_1 j_3 j j_2;\, j_{13} j_{23}) \,|\, j_1,\, j_3 j_2 (j_{23}) : jm \rangle_{qq} =$$

$$= \sum_{j_{23}} U\,(j_1 j_3 j j_2;\, j_{13} j_{23})\, \lambda_{j_{23} q}^{j_3 j_2} \,|\, j_1,\, j_2 j_3 (j_{23}) : jm \rangle_{qq} . \tag{57}$$

It follows from here that

$$\langle R_q^{32} [j_1 j_3 (j_{13}'),\, j_2 : jm] \,|\, R_q^{32} [j_1 j_3 (j_{13}),\, j_2 : jm] \rangle_{qq} \equiv I_q\,(j_{13}', j_{13}) =$$

$$= \sum_{j_{23}} U\,(j_1 j_3 j j_2;\, j_{13}' j_{23})\, U\,(j_1 j_3 j j_2;\, j_{13} j_{23})\, q^{2c(j_{23}) - 2c(j_2) - 2c(j_3)} . \tag{58}$$

As a result, instead of usual orthonormality relation for Racah coefficients of the second kind we obtain a relation between Racah coefficients of the first and of the second kinds:

$$\sum_{j_{12}} U \begin{pmatrix} j_1 & j_{13} & \\ & & ; \; j_3 j_2 q \\ j_{12} & j & \end{pmatrix} U \begin{pmatrix} j_1 & j'_{13} & \\ & & ; \; j_3 j_2 q \\ j_{12} & j & \end{pmatrix} = \sum_{j_{23}} U\,(j_1 j_3 j j_2;\, j_{13} j_{23})_q \times$$

$$\times U\,(j_1 j_3 j : j_2;\, j'_{13} j_{23})_q \, q^{2c(j_{23})-2c(j_2)-2c(j_3)} \,.$$

For example, it is easy to verify by direct calculations that for vectors $|\,1/2\;1/2\;\times\;\times(j_{13}),\;1/2\;:\;1/2\,m\rangle_{qq}$ with $j_{13} = 0,\,1$ the matrix of overlapping (58) is as follows

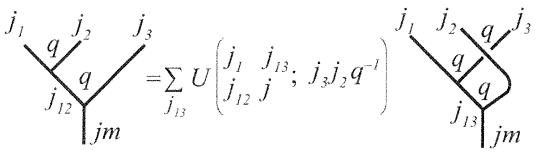
$$I_q(0,0) = \frac{[4]}{[2]\;[2]}\,; \quad I_q(1,0) = I_q(0,1) = \frac{\sqrt{(3)}}{[2]}\,(1-q^{-2})\,; \quad I_q(1,1) = \frac{q^{-2}\,[4]}{[2]\;[2]}\,.$$

Whence it is clear that the equality $I_q(j'_{13}, j_{13}) \to \delta_{j'_{13} j_{13}}$ is necessarily valid in the limit $q \to 1$. However, for arbitrary value of $q$ vectors (53) are not orthonormal, and this makes somewhat complicated the finding of matrix for the transition inverse to (53), (54). Nevertheless, this problem can be solved enough simply. There is valid the equality [2]

$$|\,j_1 j_2 (j_{12}),\, j_3 : j m\rangle_{qq} =$$

$$= \sum_{j_{13}} U \begin{pmatrix} j_2 & j_{13} & \\ & & ; j_3 j_2 q^{-1} \\ j_{12} & j & \end{pmatrix} R_q^{32} |\,j_1 j_3 (j_{13}), j_2 : j m\rangle_{qq} \qquad (59)$$

or graphically




(60)

Combining (53) and (59), we obtain a relation for Racah coefficients of the second kind which is more close to the orthonormality property

$$\sum_{j_{13}} U \begin{pmatrix} j_1 & j_{13} \\ & ; j_3 j_2 q^{-1} \\ j_{12} & j \end{pmatrix} U \begin{pmatrix} j_1 & j_{13} \\ & ; j_3 j_2 q \\ j'_{12} & j \end{pmatrix} = \delta_{j_{12} j'_{12}} \,. \tag{61}$$

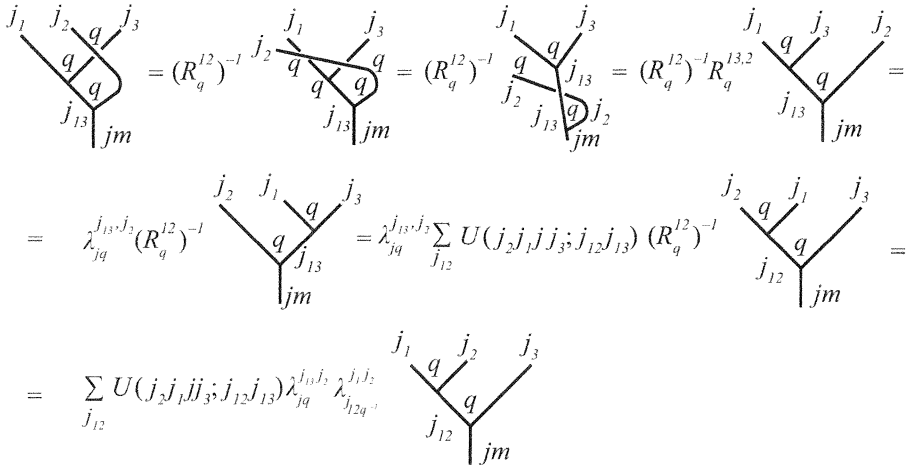
To prove the properties (59) and (61), it is necessary to express Racah coefficients of the second kind in terms of Racah coefficients of the first kind, for which the conditions of orthonormality are valid. This can be done in two ways [2]. The first way consists of the continuation of the transformation (52) and transition in the right side of this relation to the vectors $|\, j_1 j_2 (j_{12}), j_3 : jm \rangle_{qq}$:

$$R_q^{32} \,|\, j_1 j_3 (j_{13}), j_2 : jm \rangle_{qq} = \sum_{j_{23} j_{12}} U\,(j_1 j_3 j j_2;\, j_{13} j_{23}) \times$$
$$\times \lambda_{j_{23} q}^{j_3 j_2} U\,(j_1 j_2 j j_3;\, j_{12} j_{23}) \,|\, j_1 j_2 (j_{12}),\, j_3 : jm \rangle_{qq} \,. \tag{62}$$

Comparing this expression with the definition (53) of Racah coefficients of the second kind, we obtain that

$$U \begin{pmatrix} j_1 & j_{13} \\ & ; j_3 j_2 q \\ j_{12} & j \end{pmatrix} = \sum_{j_{23}} U\,(j_1 j_3 j j_2;\, j_{13} j_{23}) \lambda_{j_{23} q}^{j_3 j_2} U\,(j_1 j_2 j j_3;\, j_{12} j_{23}) \,. \tag{63}$$

The second way of reduction of Racah coefficients of the second kind to Racah coefficients of the first kind is carried out as follows:




(64)

Here we have done the following transformations:

1) have used the procedure of disinterwining of the branches $j_2$ and $j_1$ [2];

2) have applied the relation (47) to the branching $j_1 j_3$ ($j_{13}$);

3) have employed the relation (35) for matrix $R_q^{13,2}$;

4) have made standard replantation of the sprout $j_1$ from the branch $j_3$ to the branch $j_2$;

5) have applied the relations (4) and (35).

As a result, we obtain for Racah coefficients of the second kind the relation

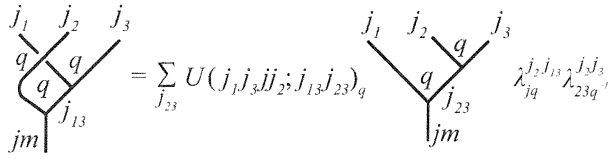
$$U\begin{pmatrix} j_1 & j_{13} & \\ & & ; j_3 j_2 q \\ j_{12} & j_1 & \end{pmatrix} = U\,(j_2 j_1 j j_3;\ j_{12} j_{13})_q \lambda_{jq}^{j_{13} j_2}\ \lambda_{j_{12} q^{-1}}^{j_1 j_2}\,. \qquad (65)$$

The meaning of all calculations performed above is, first, that from (63) and (65) we derive the identity for Racah coefficients of the first kind

$$\sum_{j_{23}} (-1)^{j_{23}-j_3-j_2} q^{c(j_{23})-c(j_2)-c(j_3)} U\,(j_1 j_3 j j_2;\ j_{13} j_{23})_q\, U\,(j_1 j_2 j j_3;\ j_{12} j_{23})_q =$$
$$= (-1)^{j-j_{13}-j_{12}+j_1} q^{c(j)-c(j_{13})-c(j_{12})+c(j_1)} U\,(j_2 j_1 j j_3;\ j_{12} j_{23})_q\,, \qquad (66)$$

second, substituting (65) in the left side of (61) and using the invariance of Racah coefficients of the first kind in respect to the substitution $q \to q^{-1}$ and their property of orthonormality, we can verify that relations (60) and (61) are valid.

Quite similarly can be written the relation




(67)

We see that the possibility of transition from one tree to another, using the procedure of intertwining and dis-intertwining branches, discussed in this paragraph at the example of trees with three dangling branches, indicates the route for obtaining the additional identities for $6j$-symbols of algebra $su_q(2)$, which cannot be found using only standard re-plantations (14). In

fact, proceeding in (66) from coefficients $U$ to $6j$-symbols, we obtain that

$$\sum_{j_{23}} [2j_{23}+1]\,(-1)^{j_{12}+j_{13}+j_{23}}\, q^{c(j_{23})-c(j_2)-c(j_3)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q \times$$

$$\times \begin{Bmatrix} j_2 & j_3 & j_{23} \\ j_1 & j & j_{13} \end{Bmatrix}_q = q^{c(j)-c(j_{13})-c(j_{12})+c(j_1)} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix}_q , \qquad (68)$$

which in the limit $q \to 1$ turns in the well-known in quantum theory of angular momentum identity for $6j$-symbols (see [2]). The value of the conception of $R$-matrix reveals itself more profound at the example with the construction of $q$-analogue for $9j$-symbol.

## 5. ACKNOWLEDGEMENT

The authors are especially grateful to the Austrian Academy of Sciences and the Russian Foundation for Fundamental Research which in the framework of the collaboration with the National Academy of Sciences of Ukraine co-financed this research.

# REFERENCES

1. G.I. Kuznetsov, S.S. Moskaliuk, Yu.F. Smirnov, and V.P. Shelest, *Graphical Theory of Representations of Orthogonal and Unitary Groups and its Physical Applications.* (Naukova Dumka Publishers, Kyiv, 1992).

2. N.A. Gromov, S.S. Moskaliuk, and Yu.F. Smirnov, *The Tree, Contractions and Analytical Continuations Methods in Representations Theory of Classical Groups and Quantum Algebras* (Naukova Dumka Publishers, Kyiv, 1993).

3. G.I. Kuznetsov, S.S. Moskaliuk, Yu.F. Smirnov, In *Proc. of the XIth International Gutsulian Workshop "Group Theoretical Methods in Physics"* (Hadronic Press, Palm Harbor, 1995). p. 23.

4. G.I. Kuznetsov and S.S. Moskaliuk, Hadronic J. Suppl. **15**(4), 349 (2000).

5. G.I. Kuznetsov and S.S. Moskaliuk Hadronic J. **24**, 17 (2001).

6. S.S. Moskaliuk, Yu.F. Smirnov Hadronic J. **24**, 129 (2001).

7. S.S. Moskaliuk, Yu.F. Smirnov Hadronic J. **27**, 399 (2004).

8. M. Moshinsky and C. Quesne J. Math. Phys, **11**, 1631 (1970).

9. Saunders Mac Lane, *Categories for the Working Mathematician* (Springer Publishers, 1998).

10. S. S. Moskaliuk and A. T. Vlassov, In *Proceedings of the 5th Wigner Symposium*, (World Sci., Singapore, 1998) p. 162.

11. S. S. Moskaliuk and A. T. Vlassov, Ukr. J. Phys. **43**, 162 (1998).

12. S. S. Moskaliuk, *From Cayley-Klein Groups to Categories* (TIMPANI Publishers, Kyiv, 2006).

ALGEBRAS GROUPS AND GEOMETRIES 29, 275 - 302 (2012) 

# FINSLER GEOMETRY IN THE PRESENCE OF ISOTOPIC FIELD CHARGES APPLIED FOR GRAVITY[1]

**György Darvas**
Symmetrion and Institute for Research Organization of the Hungarian Academy of Sciences
18 Nádor St., Budapest, H-1051 Hungary
darvasg@iif.hu

Received August 10, 2012

## Abstract

The paper specifies the conservation of the isotopic field-charge spin on the gravitational interaction, and discusses one of the consequences. First, the isotopic field-charges of the gravitational field will be defined, followed by a short presentation how the conservation of the isotopic field-charge spin has been derived. It will be shown that in the presence of a kinetic gauge field the metric of the gravitational field and its curvature should follow a Finsler geometry, that means in the presence of an isotopic mass field, the metric and the curvature depend also on velocity. In particular, the $g_{\mu\nu}$ metric tensor, and consequently the affine connection field and the curvature tensor formed from its derivatives, depend on space-time plus velocity co-ordinates. We insert this metric in the formula of the affine connection field, and the Ricci tensor. These extended formulas will be applied to one side of the Einstein equation, while on the other side there appears the stress-energy tensor specified in the presence of an isotopic mass field. We conclude that due to the velocity dependence, the Schwarzschild solution cannot be applied! It must be replaced by a Finslerian solution. Finally, as an example, we will predict that this solution may give a more accurate calculation for the precession of the perihelion of Mercury.

General Theory of Relativity was based in a significant part on the equivalence principle. This principle states equivalence between the

[1]Based on invited talks given at the 5th Petrov International Symposium on *High Energy Physics, Cosmology and Gravity* (April 29–May 5, 2012, Kyiv, Ukraine), which were partially supported by the Project No. 1202.094-12 of the Central European Initiative Cooperation Fund.

Copyright © 2012 by Hadronic Press Inc., Palm Harbor, FL 34682, U.S.A.

mass of gravity and the mass of inertia. At the same time, this statement declares that gravitational and inertial masses are not identical, for equivalence can be observed between non-identical physical properties. Papers [1]-[12] dealt in detail with this ambiguity. It is analysed in [13].

[14]-[15] formulated an assumption that the field charges of the gravitational field – gravitational and inertial are not only equivalent in their measured quantity, there exists an invariance between them. This means, that there is a symmetry according to which they are interchangeable. They can be considered as macroscopically indistinguishable physical properties of matter, which behave as isotopes of each other. Since they are qualitatively not identical, we have the right to distinguish them in our physical equations, although we can calculate with them like with equivalent value quantities. As shown in [13]-[15], mass of gravity is associated with the potential part $V$ of an object's Hamiltonian and mass of inertia is associated with the kinetic part $T$ of an object's Hamiltonian. According to this observation we can call the mass of gravity as (scalar) potential mass, and the mass of inertia as kinetic mass.

For kinetic mass is associated with (and, according to STR, depending on) the velocity of a massive object in a given reference frame, it can be described in a velocity dependent (kinetic) field, while gravitational mass belongs to a (scalar) field depending solely on the space-time co-ordinates. [16]-[17] proved the mathematical existence of a gauge invariance in a velocity dependent gauge field. This mathematical derivation led to two conserved Noether currents that exist simultaneously. This result [14]-[15] predicted at first, a conserved quantity - called isotopic field charge spin - and, at second, the exchange of two gauge quanta (bosons) between interacting mass units. One of them can be identified with the earlier assumed graviton. The other boson - let's call it 'dion' [14] - which appears in the equations due to the consideration of a velocity dependent gauge field, is new, and is a consequence of the conservation of the isotopic field charge spin.

The paper presents how did the equivalence principle applied in GTR lead to the assumption of the isotopic field charge spin and its conservation, and to the prediction of an additional boson exchange.

The invariance between isotopic field charges in the presence of a velocity dependent gauge field, and the conservation of the isotopic field charge spin were extended to the field charges of other physical interaction fields [14]-[15], since the mathematical proof [16]-[17] allowed general interacting kinetic gauge fields. This predicted the exchange of additional gauge bosons in electroweak and strong interactions as well. The result is part of the 'new physics' expected for many years in high energy physics [18]-[22], and is a candidate to replace the SUSY assumption. The difference between SUSY and the isotopic field charge spin assumption is that the former renders fermion-boson pairs as new-born brothers to each other, while the latter does fermion-fermion and boson-boson twins. There are only the boson twins new and to be discovered, since the twin brothers of fermions originate in splitting the existing ones and are assumed to be identified with the long 'known' pairs defined by the equivalence principle.

# INTRODUCTION

This paper treats fundamental physical interactions starting from two preliminary assumptions.

i) Although *mass of gravity* and *mass of inertia* are *equivalent* quantities in their measured values, they are *qualitatively not identical* physical entities. We take into consideration this difference in our equations. Then this *'equivalence is not identity' principle* is extended to sources of further fundamental interaction fields, other than gravity.

ii) *Physical interactions occur between these* qualitatively different *entities.*

These two assumptions do not contradict to any known physical theory, however, they allow another interpretation of facts built in our explanations of physical experience. Based on them we demonstrate the existence of an invariance between the two isotopic forms of the field charges, and formulate certain consequences in our view on the physical structure of matter.

# 1 THE NOTION OF ISOTOPIC FIELD CHARGES AND THEIR DYNAMICS

## 1.1 Equivalence does not mean identity

In a strict sense, *identical objects cannot be equivalent.* Only *qualitatively different* objects can be compared to conclude a quantitative equivalence between them. Equivalence always presumes the existence of at least one property, in which the compared objects differ. (Isotopic spin is a good example how to avoid ambivalence.)

The equivalence principle is one of the main pillars of the general theory of relativity (GTR). It states the equivalence of the gravitational and inertial masses. Let's consider the mass of gravity and the mass of inertia as two different properties of matter. For the same massive object can behave once as a source of gravity, then as a measure of inertia, we will imagine them as two isotopic states of the same property, called mass of the object.

As much as the mass is the field charge of the gravitational field, we will call its two isotopic states as *isotopic field charges* for the *gravitational interaction.* The gravitational mass is associated with the (scalar) potential part of that interaction, while the inertial mass with the kinetic part. In GTR the latter is attributed to the momentum densities, while the former is associated with the gravitational field energy. They are separated within the stress-energy tensor ($T_{\mu\nu}$), but according to the general relativity principle they can be transformed into each other; - we should add, at least in their quantitatively equivalent values. GTR does not make any statement about the qualitative transformation of the two kinds of masses into each other. This was a reason to identify them. The need for a qualitative transformation simply has not emerged. Nevertheless, we show that it cannot be avoided. So, we introduce distinction between masses of gravity and inertia in our equations. (In a similar way, the electric charge - i.e., the source of the electromagnetic field - is the field charge of the electromagnetic interaction; flavour and lepton charge - are the sources of the weak field; the colour charge - i.e., the source of the strong field - is the field charge of the strong interaction.) The sources - field charges - are assumed to be realised in the matter field, while they serve as sources for gauge fields. Are they really the same, or can one distinguish the two agents? The mass of gravity and the mass of inertia are considered as two equivalent quantity *isotopic states* of the field charge of the gravitational field. They represent two different qualities. Their concepts express two properties of matter, whose existence originates in different experiences. Physics established quantitative relations between them (i.e., equal values), however this fact does not vanish their qualitative difference. We argue that we have all reason to make distinction between them in our theories.

When we introduce the two isotopic filed charges in our equations, they will destroy certain symmetries of those equations. This contradicts to our experience. Therefore, there must be an invariance that compensates and restores the spoiled symmetry. To avoid the contradiction between experience and theory, we assume that the two kinds of charges of the gravitational field, should be transformed into each other by a gauge transformation. Such a gauge transformation should involve the existence of a conserved property that we define in the following way.

Since the required transformation affects the *isotopic states* of the individual *field-charges* (we mark it with ד ['dalet' the fourth letter of the Hebrew alphabet]), this transformation must be performed in a special gauge field; and

since these states can occupy two positions in that gauge field, it must be a *spin-like property*, therefore, we will call this property as *Isotopic Field-Charge Spin (IFCS)* and denote it by $\Delta$, and we will refer to the invariance transformation what we are seeking for as *isotopic field-charge gauge transformation.* This assumption assumes the existence of a local gauge field, in which the isotopic field-charge spin can rotate and occupy two states and concludes a conserved (non-Abelian) current and a corresponding class of $SU(2)$ type invariances.

For the same object can behave, e.g., in the gravitational field, once as the source of a gravitational force, and in another frame of reference as a source of a (kinetic) inertial force (cf., covariance principle), they must be able to get transformed into each other. Non-Abelian character and arbitrariness involve that the orientation of the isotopic field-charge spin is of no physical significance. If we determine the proper form of this invariance transformation, it will counteract the loss of symmetry between the two kinds of field-charges, and bring our equations in compliance with the experimental observations.

The required invariance shows certain formal similarities to YM-type invariances [23]-[24]. However, it must differ from them in at least two features. Once, the concerned physical property, namely the isotopic field charge (IFC, ד), is a quite different physical property than the isotopic states of nucleons. Secondly, the gauge field, and consequently the gauge transformation that rotates the isotopic field charge spin (IFCS, $\Delta$) in this gauge field, are quite different from the isotopic gauge field derived for the isotopic spin transformation. (For specification, see section 2.)

The existence of such an invariance transformation provides us with a symmetry, and consequently with a conservation law, with the conservation of

the introduced new property ($\Delta$) of the field-charges. The conservation of isotopic field-charge spin is identical with the requirement of invariance of all interactions under isotopic field-charge spin rotation (in the gauge field where it is interpreted). Accordingly, all physical interactions should be invariant under a transformation in a specific gauge field, more precisely, under a rotation of the property, called isotopic field-charge spin ($\Delta$). [15]-[17] proved that invariance transformation.

### 1.2 Interaction between the isotopic field charges

When we take a measure on an object, we have no experience that we found it in one or the opposite isotopic state. Would we observe a single particle, it were either in one or in the other IFCS state. We can call the two states as potential and kinetic, scalar and vector, or bound and free states. However, our measurement records a mixture of the two states. Nevertheless, we do not observe the individual IFCS states. Our observation suggests that they behave as being in both states, each measured object can occupy both a potential (bound) and a kinetic (free) IFCS state. In the lack of experience to catch a particle in one or the other stable state, we have good reason to assume that they permanently change their states. (Randomly or with a stable frequency, they may probably follow a similar mechanism like quarks do during their colour change via gluon exchange).

Let us consider a model of a doublet, when a particle can be in a potential state ($V$) and in a kinetic state ($T$). According to its actual state it has potential or kinetic energy respectively. According to our observation all particles possess both. We can interpret the phenomenon in the following way: In a *probabilistic model* we can consider that the wave function of the given particle may be in a potential state with amplitude $\psi_T$, or in a kinetic state with amplitude $\psi_T$. We detect a probabilistic mixture in a measurement. In a large set of particles (e.g., in the case of a massive body consisting of many particles) the probabilities reach a stable proportion and we observe stabilised measurable potential and kinetic energies in a given reference frame. A *harmonic oscillator model* presumes the permanent change of a single particle between its two isotopic field charge states. A particle *in a potential state* plays the role of the *source of a scalar field.*

Therefore a potential isotopic field charge (we denote by $\urcorner_V$) is a scalar quantity. A particle *in a kinetic state* serves as a *current source of a vector field.* So a kinetic isotopic field charge (we denote by $\urcorner_T$) plays roles in three vector components according to three, directed, independent components of a field charge current. An important consequence of the switch between the two IFC states is that the isotopic field charges must commute between a scalar and three components of a vector quantity, according to the velocity components of the kinetic state in the given reference frame.

### 1.3 Isotopic field charges in the gravitational field

As a consequence of the distinction between $m_V$ and $m_T$, as well as the association of the energy content with the mass $m_V$ and the components of the momentum with $m_T$, we lose also the symmetry of the $T_{\mu\nu}$ energy-momentum tensor. To retain symmetry in Einstein's field equations we must require again the invariant transformation of $m_V$ and $m_T$ into each other in an appropriate gauge field. We refer to Mills [24] who foresaw the possible generalisation of YM type gauge invariance in general relativity "in close analogy with the curvature tensor". If we consider the energy-momentum tensor (in which both isotopic states of mass appear) as the source of the gravitational field, then - in the usual way - a scalar and a vector potential can be separated. (A hypothetic vector potential is justified by a non-static effect, e.g., acceleration, in the field.) Although, unlike QED, there is no analogy with the meaning of a vector potential of the electromagnetic field, the consideration of the kinetic (inertial) mass as an individual physical property against the gravitational mass may lend certain meaning to a gravitational vector potential. We can explain this so, that $m_4$ in $T_{44}$ does not compose a fourth component of a four-vector in the classical theory of gravitation where there is a single scalar mass, while if we consider now $m_4 = m_V$, the three components of the kinetic mass $m_T$ can compose a three-vector, however $T_{i4}$ will not form a four vector either.

To maintain the Lorentz invariance of our physical equations in the gravitational field, we must demand to restore the invariance of $\begin{pmatrix} \vec{m}_T \\ m_V \end{pmatrix}$ under an *additional transformation* that should *counteract the loss of symmetry*

*caused by the introduction of two isotopic states of mass.* We discuss that transformation in section 2. Further, in the case of gravitation the relation of the scalar and the vector fields are not linear even if we have not made distinction between the potential and kinetic masses. The non-linearity is coded in the relation of the tensors [25] at the left side of the Einstein equation (in units $c = 1$),

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

or $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ where the Einstein tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ whose covariant derivative must vanish.

Since our $T_{\mu\nu}$ tensor on the right side has already lost its symmetry, we can take $\Lambda g_{\mu\nu}$ into account within a modified $T'_{\mu\nu}$ - handling the gravitational and kinetic masses in it together with the dark energy - and we get the following formally symmetric equation:

$$G_{\mu\nu} = 8\pi G T'_{\mu\nu}\,.$$

(The disadvantage of this apparently quasi-symmetric form is that the metric tensor $g_{\mu\nu}$ appears in the expressions at both sides of the equation.) It is only our enigmatic hope that the asymmetry hidden inside $T'_{\mu\nu}$ will be restored with the conservation of the IFCS for the isotopic gravitational field charges together with the dark energy. Nevertheless, even if the latter fails, the symmetry of the energy-momentum tensor can be saved by the invariant gauge transformation of the IFCS. The most important analogy is between the behaviour of the potential and the kinetic field charges of the individual fields that makes probable to conjecture that a unique transformation will assure their invariance (cf., section 2).[2]

(See in details in sections 3-4.)

---

[2]We must add to the conjecture of the "unique" transformation a few remarks. As [26] stated, "In contrast to the symmetry or invariance requirement in STR, the principle in GTR is most often presented as strictly speaking a covariance requirement." Gauge theories behave like GTR, at least in this respect. General covariance "is not tied to any geometrical regularity of the underlying spacetime, but rather the form invariance (covariance) of laws under arbitrary smooth coordinate transformations" [26, p. 34]. Weyl [27] found that the more general geometry resulting from admitting local changes called gauges described not only gravity but also electromagnetism. He showed also that the

# 2. CONSERVATION OF THE ISOTOPIC FIELD CHARGE SPIN ($\Delta$)

Distorted symmetry of our equations[3] - what is not in accordance with experience - can be restored by proving that there exists an invariance between the twin brothers of the field charges (sources of the fields) split according to the introduced new property ($\Delta$). Invariance means that particles, disposed with these properties, can be exchanged. The "exchange rate" (gauge) depends on the velocity of the kinetic field charge compared to the respective matter field (i.e., to the scalar potential field charge in rest in that field). The validity of the assumption can be verified by demonstrating the existence of the gauge bosons that mediate the exchange. This invariance as soon as proven means a new symmetry principle of nature. This perspective is challenging!

Section 2.2 below presents the main lines of the mathematical proof [17] of such invariance. The demonstration of the predicted gauge bosons is left to the experience.

## 2.1 Velocity dependent phenomena

We know certain phenomena in classical physics that depend on velocity in a given reference frame. As examples, there can be mentioned first the kinetic energy, then the Lorentz force, and the covariant effect of the Lorentz transformation:

[$(x^\mu)' = \Lambda^\mu_\nu(\nu)x^\nu$ for space-time vectors, and $(F^{\mu\nu})' = \Lambda^\mu_\alpha(\nu)\Lambda^\nu_\beta(\nu)F^{\alpha\beta}$ for the electro-magnetic field tensor].

---

conservation laws of Noether follow in two distinct ways in theories with local symmetries. This led to the Bianchi identities, which hold between the coupled equations of motion, and which are due to the local gauge invariance of action. Later [28] demonstrated that the conservation of the electric charge followed from the local gauge invariance in the same way as does energy-momentum conservation from co-ordinate invariance in GTR.

[3]According to Higgs [29]: "The idea that the apparently approximate nature of the internal symmetries of elementary-particle physics is the result of asymmetries in the stable solutions of exactly symmetric dynamical equations, rather than an indication of asymmetry in the equations themselves, is an attractive one." Please, compare this notice with Wigners concern [30]!

Descriptions of the mentioned phenomena handle the space-time co-ordinates as indirect variables. The Lorentz invariance depends only on the velocity difference between the compared systems. In general, kinetic quantities depend first on velocity in the chosen reference frame, and only indirectly, through $\nu = \nu(x_i, t)$ on the space-time variables. As [24] observed, "Hamilton's principle was first discovered in connection with mechanical systems, where the Lagrangian turns out to be the difference between the kinetic and potential energies, but the principle is easily extended to include velocity-dependent forces of certain types", including, e.g., the magnetic force on a moving, electrically charged particle.

It is not surprising that phenomena related solely to the kinetic part of the Hamiltonian ($T$) can be described in a velocity dependent, i.e., kinetic field $D_T = D[\nu(x_i, t)]$ where the dependence on the local co-ordinates is indirect. This does not disclose the possibility of localisation of the theory in space-time, however, it does not ensure it automatically. Local symmetry in a kinetic field means that the objects, fields or physical laws in question are invariant under a local transformation, namely under a set of continuously infinite number of separate transformations with an arbitrarily different one at every velocity in the given reference frame.

The isotopic field charge (IFC, $\urcorner$) as a property can be identified in the case of the gravitational field with the properties of the masses of gravity and inertia respectively. The potential isotope of $\urcorner(\urcorner_V)$ depends directly on space-time co-ordinates. The physical state of the kinetic isotope of $\urcorner(\urcorner_T)$ depends primarily on the components of its velocity (and indirectly on its space-time co-ordinates). When we try to specify physical phenomena that distinguish kinetic behaviour of objects from their behaviour in a field caused by another, potential source (i.e., $\urcorner_V$) we should make attempt to seek for a description in a velocity dependent field.

## 2.2 Mathematical background of the conservation of $\Delta$

For the sake of the description of the mentioned distinction, we introduce a gauge field $D_\mu$, that depends primarily on velocity. We derived a set of conserved currents in such a field [17]. The mathematical treatment is as much general as possible, while we made a specification. Namely, Noethers

second theorem allows the dependence of the concerned fields (on which the Lagrangian depends) on any, general co-ordinate. Certain physical theories restrict themselves on the four space-time co-ordinates as dependent variables. We discuss fields that depend on co-ordinates in the velocity four-space, (and handle the space-time co-ordinates as indirect variables).

For the effects of a general non-Abelian group on the local gauge invariance are to be described, we refer to the [24] review paper. We partially use the methods of his description of YM type gauge fields. We introduce a new type of localised gauge field that does not coincide with the isotopic spins YM field, marked by **B** in [23] and [24]; this field, marked by **D**, is *per defintionem* different from the YM field.[4] In our discussion, the **D** gauge field, introduced below, depends directly on the velocity-space coordinates, while the matter field depends directly on the four dimensional space-time co-ordinates. In other words, this means that although we primarily use coordinates of the velocity-space, our derivations are indirect and include derivatives with respect to the space-time co-ordinates (cf., the introduction of the relativistic $\lambda_{\mu}^{\nu}$ tensor below) and play important role in our conclusions. This is an expression of the facts that we observe the physical events (occurring even in the velocity space) with respect to the 4D space-time, on the one hand, and that our operators should effect complex $\psi(x_{\nu})$ fields which depend on the four space-time co-ordinates, on the other.

We extend the role of the co-ordinates to a set of generalised variables alike Noether [31] did. These variables may be the four space-time co-ordinates or they may be others (and their number may vary). In her mathematical terms of invariant variational problems, the space-time co-ordinates did not play a distinguished role. According to her second theorem, other variables, among others (e.g., velocity-space co-ordinates), are allowed which may implicitly depend on the space-time co-ordinates. For practical reasons we replace the $f(\dot{x}_{\mu}, x_{\nu})$ dependence with a $f(\dot{x}_{\mu}(x_{\nu}))$ dependence. The localisation is present here too (in the above generalised, Noetherian sense), although it makes us possible another way of calculating it.

[4] Although we use the letter "**D**" to denote this gauge field, in [24] and many other publications that letter denoted the covariant derivative, which we will mark by careted (capped) derivation mark $\hat{\partial}$.

We were seeking for invariance between scalar fields and (gauge) vector fields that describe kinetic processes, the latter depending therefore primarily on velocity. For this reason, we consider Lagrangians which depend on matter fields $\varphi_k$, and gauge fields $D_{\dot{\mu},\alpha}$, which all depend - in simple mathematical terms - on parameters. In physical terms these parameters are generally identified with the four space-time co-ordinates. In our specific case the dependence of $\mathbf{D}$ on $x_\mu$ will be given by the formula: $D_{\dot{\mu}} = D_\mu \left(\frac{\partial x^\mu}{\partial x_4}\right)$, or in another form $D_{\dot{\mu}} = D_\mu \left[\dot{x}^\mu(x_\nu)\right]$. The 2nd theorem of Noether is just about Lagrangians, which depend on arbitrary number of fields with arbitrary finite number of derivatives by arbitrary number of parameters. We can apply her theorem here because in mathematical terms she did not specify either the physical-mathematical character or the number of applicable parameters. Our consideration will be justified by the final result, which demonstrates that in a boundary situation, namely in the absence of a velocity-dependent gauge field we obtain the same currents that were derived in a space-time dependent field, (cf. Eqs. (4) and (7) below). In other words, in the absence of relativistically high velocities or acceleration, the effect of the velocity dependent gauge field can be neglected, and we get back to the same currents as derived in the semi-classical, only space-time dependent gauge's case. At the same time, in the presence of a velocity dependent gauge field, we derived new conserved Noether currents [17].

### 2.2.1 Noether's currents for gauge invariance localised in the velocity space

The presentation discusses general, non-Abelian case. Let's first introduce a (kinetic) $\mathbf{D}$ field localised in the velocity space, with components $D_{\dot{\mu}} = D_{\dot{\mu}}(\dot{x}^\mu)$, where $\dot{x}^\mu = \dot{x}^\mu(x_\nu)$; $(\mu, \nu = 1, 2, 3, 4)$; (dotted indices denote the velocity-space components). We introduce a $\lambda^\nu_\mu$ tensor defined as $\lambda^\nu_\mu = \partial_\mu \dot{x}^\nu = \frac{\partial \dot{x}^\nu}{\partial x_\mu}$ (Lorentz invariant acceleration), which characterises the changes of the velocity-space components in the space-time. Localisation will be taken into consideration in this way (we refer to the generalised interpretation of localisation as defined above).[5]

[5]Relativistic covariance under Lorentz transformation $S(\Lambda)$ and its consequences are a standard part of quantum field theory textbooks for long, e.g., [32, Sec. 2.1.3]. Here

In general, we base on a transformation group $G$ and the transformations of its elements into each other $T\left[G_{\infty,\rho}\right] = T\left[p_\alpha(x_\beta)\right]$, where the number of parameters are arbitrary finite numbers $(\alpha = 1, \ldots, \rho))$; , $(\beta = 1, \ldots, \sigma)$. The $p$ are parameters on which the transformations, constituting the group elements, depend. They take the form of functions $p_\alpha(x_\beta)$ and their derivatives. The group transformations depend on $p$ and are finitely differentiable. $G$ may take the form of different groups, depending on the concrete form of interaction in subject, namely $SO(3,1)$, $U(1)$, $SU(2)$, $SU(3)$ in the cases of the fundamental physical interactions.

We consider a Lagrangian density $L(\varphi_k, D_{\dot{\mu},\alpha})$, where $\varphi_k$, $(k = 1, \ldots, n)$ are the matter fields - which also includes the velocity field $\dot{x}^\mu = \dot{x}^\mu(x_\nu)$ –, and $D_{\dot{\mu},\alpha}$, $(\alpha = 1, \ldots, N)$, are the (kinetic) gauge fields. We assume, that $L(\varphi_k, D_{\dot{\mu},\alpha})$ is invariant under the local transformations of a compact, simple Lie group $G$ generated by $T_\alpha$, $(\alpha = 1, \ldots, N)$, where $[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta}T_\gamma$, and $CY^\gamma_{\alpha\beta}$ are the so-called structure constants, corresponding to the actually considered individual physical interactions symmetry group.[6] For examples, in the case of $SU(2)$ symmetry, $G$ consists of $2 \times 2$ matrices with 3 independent components, representing a state doublet, and in the case of $SU(3)$ its matrix has 8 independent components, representing a state triplet. For simplicity we assume that the matter fields belong to a single, $n$-dimensional representation of $G$.

Let us consider a local transformation $V(\dot{x}) \in G$ parameterised by $p_\alpha(\dot{x})$ that acts on $\psi$ as $\psi = V\psi'$

$$V(\dot{x}) = e^{-ip_\alpha(\dot{x})T_\alpha}$$

The infinitesimal transformations of the matter- and the gauge fields determine the covariant derivatives of $\psi$ in the gauge field. (For invariance, we can require that the derivatives of $\psi$ coincide with the derivatives of $V\psi'$). The infinitesimal transformations can be formulated as follows:

$$\delta\varphi_k = -ip_\alpha(\dot{x})(T_\alpha)_{kl}\varphi_l(\dot{x}) \qquad (k = 1, \ldots, n)\,, \tag{1}$$

---

we take into account time derivatives of Lorentz transformed velocities.

[6] We partly follow the clues by Higgs [29] and Weinberg [33] at the beginning of their papers with the exception that we consider different dependencies in the potential and kinetic Hamiltonian terms.

where the $T_\alpha$ are matrix-representation operators generating the group $G$, with the above commutation rule $[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta}T_\gamma$, and

$$\delta D_{\mu,\alpha} = \frac{1}{2}\partial^{\dot\rho}p_\alpha(\dot x)\partial_\mu \dot x^\rho + C^\gamma_{\alpha\beta}p_\beta(\dot x)D_{\dot\mu,\gamma}(\dot x) \qquad (\alpha = 1,\ldots,N) \tag{2}$$

where $\partial^{\dot\rho} = \frac{\partial}{\partial \dot x^\rho}$, and ג (Hebrew $g$, gimel) denotes a general coupling constant, which can be replaced by a concrete coupling constant for each individual physical interaction.

For the induced infinitesimal transformation $\delta L$ of the Lagrangian density $L(\varphi_k, D_{\dot\mu,\alpha})$, on using the field equations for both the matter and the gauge fields, one obtains

$$\delta L = \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\varphi_k)}\delta\varphi_k + \frac{\partial L}{\partial(\partial_\mu D_{\nu,\alpha})}\delta D_{\nu,\alpha}\right). \tag{3}$$

One would like to describe the events, resulted in the interaction between the matter field and the kinetic (velocity-space dependent) gauge field, as they are observed from the usual 4D space-time. Therefore one needs to apply derivatives by the space-time co-ordinates. Substituting from (1) and (2) into (3), using the notation $\frac{\partial \dot x^\nu}{\partial x_\mu} = \partial_\mu \dot x^\nu = \lambda^\nu_\mu$ (Lorentz invariant acceleration), and a permutation of the indices, one can obtain

$$\begin{aligned}\delta L =& \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\varphi_k)}(-i)p_\alpha(\dot x)(T_\alpha)_{kl}\varphi_l(\dot x)\right) + \\ &+ \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu D_{\dot\nu,\alpha})}\frac{1}{\text{ג}}\partial^{\dot\rho}p_\alpha(\dot x)\lambda^\rho_\nu\right) + \delta_\mu\left(\frac{\partial L}{\partial(\partial_\mu D_{\dot\mu,\alpha})}C^\gamma_{\alpha\beta}p_\beta(\dot x)D_{\dot\nu,\gamma}(\dot x)\right).\end{aligned}$$

We have derived from here the following two sets of equations:

$$J^{(1)\nu}_\alpha(x) = \partial_\mu F^{(1)\mu\nu}_\alpha(x) \qquad \partial_\nu J^{(1)\nu}_\alpha = 0 \tag{4}$$

$$J^{(2)\nu}_\alpha = \partial_\mu F^{(2)\mu\nu}_\alpha \qquad \partial_\nu J^{(2)\nu}_\alpha = 0 \tag{5}$$

completed with

$$\frac{\partial L}{\partial(\partial_\mu D_{\dot\mu,\alpha})}\lambda^\rho_\nu + \frac{\partial L}{\partial(\partial_\nu D_{\mu,\alpha})}\lambda^\rho_\mu = 0 \tag{6}$$

this set (4)-(6) demonstrates, that in the presence of a kinetic (velocity-dependent) gauge field, there exist two (families of) conserved Noether currents. Although the two conserved currents are not independent, in the presence of a kinetic gauge field they exist simultaneously. (One can easily see, that $\lambda_\mu^\nu$ mixes the components of the gauge-field currents depending on the 4D velocity space in a similar way, like the Lorentz transformation mixes the co-ordinates of four-vectors in the 4D space-time; since the $\lambda_\mu^\nu$ tensor was defined to characterise the changes of the velocity-space components - accelerations - in the space-time.) Taking into account the conditions how we have obtained these currents, one can write $J_\alpha^{(1)\mu}$ as

$$J_\alpha^{(1)\mu}(\dot{x}) = i\rlap{⊐}\frac{\partial L}{\partial(\partial_\nu, \varphi_k)}(T_\alpha)_{kl}\varphi_i(\dot{x})\,. \tag{7}$$

The most significant conclusion of the above cited derivation (cf., [17]) is that in the presence of a kinetic gauge field $\mathbf{D}$, there appear extra $J_\alpha^{(2)\nu}$ conserved currents. Taking into account conditions of the derivation of $J_\alpha^{(2)\nu}$, one can write it in the form

$$J_\alpha^{(2)\nu} = i\rlap{⊐}\left[\frac{\partial L}{\partial(\partial_\mu\varphi_k)}(T_\alpha)_{kl}\varphi_l(\dot{x})\lambda_\mu^\nu - C_{\alpha\beta}^\gamma D_{\dot{\omega},\beta}(\dot{x})\lambda_\mu^\omega \times F_\gamma^{(2)\mu\nu}(x)\right]. \tag{8}$$

Their dependence on the velocity-space gauge is apparent, although, none of the conserved vector currents involve the gauge parameters $p_\alpha(\dot{x})$ and their derivatives.

¿From (4) and (7), considering consequences of (6), one obtains

$$\partial_\mu F_\alpha^{(1)\mu\nu}(\dot{x}) = i\rlap{⊐}\frac{\partial L}{\partial(\partial_\nu\varphi_k)}(T_\alpha)_{kl}\varphi_l(\dot{x})\,. \tag{9}$$

¿From (5) and (8), considering the concrete forms of the covariant derivatives, one obtains

$$\hat{\partial}_\mu F_\alpha^{(2)\mu\nu}(x) = i\rlap{⊐}\frac{\partial L}{\partial(\partial_\mu\varphi_k)}(T_\alpha)_{kl}\varphi_l(\dot{x})\lambda_\mu^\nu\,. \tag{10}$$

### 2.3 Mathematical conclusions

*First conclusion* - of the conserved Noether current (4) - is a conserved quantity: Conservation of the field charge ($\urcorner$).

*Second conclusion* - of the conserved Noether current (5) - is another conserved quantity: Conservation of the isotopic field charge spin ($\Delta$).

Further, we could derive, in the usual way, the total isotopic field charge spin

$$\Delta = \frac{i}{\urcorner} \int J^{(2)4} \mathrm{d}^3 x \,.$$

which is independent of time and independent of Lorentz transformation. $J^{(2)\mu}$ does not transform as a vector, while $\Delta$ transforms as a vector under rotations in the isotopic field charge spin field.

### 2.4 Physical conclusions

Coupling of the two conserved quantities ($\urcorner$ and $\Delta$), what is based on the dependence of the two currents $J_\alpha^{(1)\mu}$ and $J_\alpha^{(2)\mu}$ on each other, has physical consequences. The quantities, whose conservation they represent, and which are coupled (by $\lambda_\mu^\nu = \partial_\mu \dot{x}^\nu$), exist simultaneously. The *derived conservation law verifies* just *the invariance between two isotopic states of the field charges, namely between the potential* $\urcorner_V$ *and the kinetic* $\urcorner_T$ what we intended to prove. We obtained, that in *the presence of kinetic fields we have two conserved currents that are effective simultaneously.* The kinetic gauge field $\mathbf{D}$ is present simultaneously with the interacting matter [$\varphi$] and gauge [$\mathbf{B}$] fields. The presence of $\mathbf{D}$ corresponds to the property of the field charges $\urcorner$ of the individual fields that they split in two isotopic states, and analogously to the isotopic spin, we named these two states *isotopic field charge spin* what we denoted by $\Delta$. The source of the isotopic field charge spin ($\Delta$) is the field $\varphi(\dot{x})$, in interaction with the kinetic gauge field $\mathbf{D}$.

The physical meaning of $\Delta$ can be understood, when we specify the transformation group associated with the $\mathbf{D}$ field, which describes the transformations of $\urcorner$ (i.e., the isotopic field charges). $\urcorner$ can take two (potential and kinetic) isotopic states $\urcorner_V$ and $\urcorner_T$ in a simple unitary abstract space.

Their symmetry group is $SU(2)$, that can be represented by $2 \times 2$ $T_\alpha$ matrices. There are three independent $T_\alpha$ that may transform into each other, following the rule $[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta}T_\gamma$, where the structure constants can take the values 0, $\pm 1$. Let $T_1$ and $T_2$ be those which do not commute with $T_3$; they generate transformations that mix the different values of $T_3$, while this "third" component's eigenvalues represent the members of a $\Delta$ doublet. For the isotopic field charges compose a $\urcorner$ doublet of $\urcorner_V$ and $\urcorner_T$, the field's wave function can be written as

$$\psi = \begin{pmatrix} \psi_T \\ \psi_V \end{pmatrix} . \tag{11}$$

(11) is the wave function for a single particle which may be in the "potential state", with amplitude $\psi_V$, or in the "kinetic state", with amplitude $\psi_T$. $\psi$ in (11) represents a mixture of the potential and kinetic states of the $\urcorner$, and there are $T_\alpha$ that govern the mixing of the components $\psi_V$ and $\psi_T$ in the transformation. $T_\alpha$ ($\alpha = 1, 2, 3$) are representations of operators which can be taken as the three components of the isotopic field charge spin, $\Delta_1$, $\Delta_2$, $\Delta_3$ that follow the same (non-Abelian) com-mutation rules as do the $T_\alpha$ matrices, $[\Delta_1, \Delta_2] = i\Delta_3$, etc. These operators represent the charges of the isotopic field charge spin space, and $\psi$ are the fields on which the operators of the gauge fields act.

The quanta of the **D** field should carry isotopic field charge spin $\Delta$. The $\Delta$ doublet, as a conserved quantity, is related to the two isotopic states of field charges ($\urcorner$), and the associated operators ($\Delta_i$) induce transitions from one member of the doublet to the other.

### 2.5 Interpretation of the isotopic field charge spin conservation

Invariance between $\urcorner_V$ and $\urcorner_T$ means that they can substitute for each other arbitrarily in the interaction between field charges of any given fundamental physical interaction. They appear at a probability between [0, 1] in a mixture of states in the wave function $\psi$ (11) so that the Hamiltonian of a *single particle* oscillates between $V$ and $T$, while the Hamiltonian of a *composite system* is a mixture of the oscillating components of the particles that constitute the system. The individual particles in a *two-particle system* are either in the $V$ or in the $T$ state respectively, and switch between

the two roles permanently; while the observable value of $H$ is the expected value of the mixture of the actual states of the two, always opposite state particles.

The invariance between $\urcorner_V$ and $\urcorner_T$ (what is ensured by the conservation of $\Delta$), and their ability to swap, means also that they can restore the symmetry in the physical equations which was lost when we replaced the general $\urcorner$ (in our case mass $m$) by their isotopes $\urcorner_V$ and $\urcorner_T$ (concretely $m_V$ and $m_T$).[7]

We denote the *predicted* quanta of the **D** field by $\delta$. We call this hypothetical boson *"dion"*, after the Greek term meaning 'flee', 'flight', 'rout' in English. The $\delta$ quanta (dions) carry the $\Delta$ (isotopic field charge spin as a physical property: charge of the **D** field). According to the IFCS model, gravitational interaction takes place between two massive particles wit the simultaneous exchange of a graviton and a dion.

Starting from the equivalence principle, through the qualitative distinction of the masses of gravity and inertia as isotopic field charges of the gravitational field and interaction between them, we concluded the prediction of a boson that mediates their interaction.

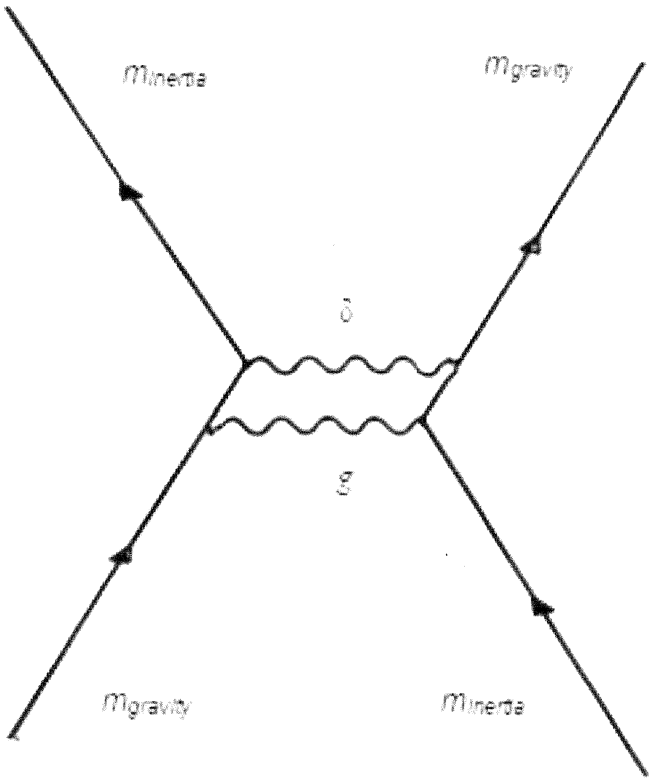
## 3 Finsler geometry in the presence of isotopic field charges

Let us specify (9) for the gravitational field [35]. The right side of the equation contains the scalar field that serves for the source of the gravitational field. The $\beth$ can be replaced by the gravitational coupling constant $g$. As we noticed, the dependence on the gauge fields is on the left side of the equation (9). $F^{(1)\mu\nu}(\dot{x})$ must satisfy the

$$T_{\mu\nu} = F_{\mu\lambda}F_{\lambda\nu} + \frac{1}{4}\delta_{\mu\nu}g^{\kappa\sigma}F_{\lambda\sigma}g^{\lambda\rho}F_{\kappa\rho}$$

identity for the energy-momentum tensor $T_{\mu\nu}$. (In order to bring this form in compliance with the indices in (9), one should raise the indices by multiplying with the metric tensor $g_{\beta_\gamma}$ in the right side.) This energy-momentum

[7] Consequences of the application of effective field theories were analysed e.g., in philosophy by E. Castellani [34] and in physics by S. Weinberg [22].




tensor $T_{\mu\nu}$ can be expressed by the way of the Einstein equation

$$T_{\mu\nu} = -\frac{1}{8\pi G_N}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}) \tag{12}$$

where $R_{\mu\nu}$ is the Ricci tensor defined by the help of the derivatives of the metric tensor $g_{\mu\dot{\nu}}$, $\mathbf{R}$ is the Ricci scalar formed from the Ricci tensor (Riemann curvature) and the metric tensor, and $\Lambda$ is a constant of Nature, as well as $G_N$ the constant of Newton.

The metric tensor $g_{\mu\nu}$ and its derivatives depend on the localisation of the given point in the space-time in the General Theory of Relativity (GTR), and are subject of Riemann geometry. In the presence of a kinetic field, that means, isotopic mass field $\mathbf{D}$ (mass being the field-charge of the gravitational

field), however, the curvature depends also on velocity. (Whose velocity? On the actual inertial velocity of atest unit-mass placed in a given space-time point in the referenceframe fixed to the source of a scalar gravitational field $\varphi$ which appears on the right side of (9).) The $g_{\mu\nu}$ metric tensor, and consequently the affine connection field and the curvature tensor formed from its derivatives, depend on space-time and velocity co-ordinates. With the appearance of the dependence on the velocity vector, the curvature becomes dependent on its direction in each space-time point. The direction (additional parameter) attributed to each space-time point is defined by the orientation of the velocity of a test unit-mass in the given space-time point, $\frac{\mathbf{v}}{|\nu|}$. The curvature can no more follow a "simple" Riemann geometry, it follows a Finsler geometry whose metric is defined by the dependence of $g_{\mu\nu}$ on ($x_\sigma$ and) $\dot{x}_\rho$.

Of course, the space-time plus four-velocity dependence of the metric tensor $g_{\mu\nu}$ affects its all derivatives, including the formation of the affine connection field (from first derivatives) and the Riemann curvature (or Ricci tensor, second, covariant derivative)

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}\left[\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right] \qquad \Gamma^\lambda_{\mu\nu} = g^{\lambda\rho}\Gamma_{\rho\mu\nu}$$

and

$$R_{\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\lambda_{\sigma\lambda}\Gamma^\sigma_{\mu\nu}\,.$$

The solution of the Einstein equation in velocity dependent field with Finsler geometry must necessarily lead to solutions different from that of Schwarzschild.

## 4 The role of the isotopic field charge spin conservation

The role of equation (12) is to retain the invariance between the two isotopic forms, namely gravitational and inertial, of masses. The importance of this is to save the covariance of our equations. Since there appear two different kinds of (isotopic) masses in the energy-momentum "four-vector" (in the fourth column of $T_{\mu\nu}$) it does no more transform as a vector, and Lorentz transformation can no more guarantee alone the covariance of our equations.

As a consequence of the distinction between $m_V$ and $m_T$, as well as the association of the energy content with the mass $m_V$ and the components

of the momentum with $m_T$, we lose also the symmetry of the $T_{\mu\nu}$ energy-momentum tensor. To retain symmetry in Einstein's field equations we must require again the invariant transformation of $m_V$ and $m_T$ into each other in an appropriate gauge field, namely in **D**. We refer to [24] who foresaw the possible generalisation of YM type gauge invariance in general relativity "in close analogy with the curvature tensor". If we consider the energy-momentum tensor (in which both isotopic states of mass appear) as the source of the gravitational field, then - in the usual way - the scalar and the vector potential can be separated. See, $m_4$ in $T_{44}$ does not compose a fourth component of a four-vector in the classical theory of gravitation where there is a single scalar mass. If we consider now $m_4 = m_V$, the three components of the kinetic mass $m_T$ can compose a three-vector, however $T_{\mu 4}$ will not form a four vector either. To maintain the Lorentz invariance of our physical equations in the gravitational field, we must demand to restore the invariance of $\begin{pmatrix} \vec{m}_T \\ m_V \end{pmatrix}$ under an *additional transformation* that should *counteract the loss of symmetry caused by the introduction of two isotopic states of mass.* We discussed that transformation in section 2. Further, in the case of gravitation the relation of the scalar and the vector fields are not linear even if we have not made distinction between the potential and kinetic masses. The non-linearity is coded in the relation of the tensors [25] at the right side of the Einstein equation (12) (in units $c = 1$), or we can write $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ where the Einstein tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ whose covariant derivative must vanish.

Since our $T_{\mu\nu}$ tensor has already lost its symmetry, we can take $\Lambda g_{\mu\nu}$ into account within a modified $T_{\mu\nu}$ - handling the gravitational and kinetic masses in it together with the dark energy - and we get the following formally symmetric equation: $G_{\mu\nu} = 8\pi G T'_{\mu\nu}$.

The symmetry of the energy-momentum tensor can be saved by the invariant gauge transformation of the IFCS. The most important analogy is between the behaviour of the potential and the kinetic field charges of the individual fields that makes probable to postulate a unique transformation to assure their invariance (cf., section 2).[8] So the invariance under the Lorentz

[8] As [26] stated, "In contrast to the symmetry or invariance requirement in STR, the principle in GTR is most often presented as strictly speaking a covariance requirement."

transformation combined with the invariance of the isotopic field charge spin field provide together the covariance of the gravitational equation. However, this combined transformation should now be taken into consideration in a field with a metric depending on all space-time and velocity co-ordinates, following a Finsler geometry.

# 5 Appendix

*Comparison of the invariance properties in classical GTR and in the IFCS model*

In classical physics, conservation laws - as consequences of the invariance properties of the investigated systems - can be obtained by integration of the Euler-Lagrange equations of motion of classical mechanical point systems. According to Hamilton's principle the variation of the action integral of the systems Lagrangian must be zero. These conservation laws include the conservation of the energy - invariance under translation in time. That conserved energy is equivalent with a well determined amount of mass $E = mc^2$, where $m = m_V$ is gravitational mass, and this conservation law does not provide any information on the quantity of kinetic mass.

In general relativistic treatment, the source of the gravitational field is the $T_{\mu\nu}$ momentum-energy stress tensor, which includes the sources of inertial and gravitational effects as well. Applying the same variational method and integration for the Einstein equation (using $[+\ +\ +\ -]$ signature) we derive the conservation of the $-T_{44}$ element of the $T_{\mu\nu}$ momentum-energy stress tensor. $-T_{44}$ is energy density of the gravitational field, and is proportional to a certain amount of mass. According to invariance under

---

Gauge theories behave like GTR, at least in this respect. General covariance "is not tied to any geometrical regularity of the underlying spacetime, but rather the form invariance (covariance) of laws under arbitrary smooth coordinate transformations" [26, p. 34]. [27] found that the more general geometry resulting from admitting local changes called gauges described not only gravity but also electromagnetism. He showed also that the conservation laws of Noether follow in two distinct ways in theories with local symmetries. This led to the Bianchi identities, which hold between the coupled equations of motion, and which are due to the local gauge invariance of action. Later [28] demonstrated that the conservation of the electric charge followed from the local gauge invariance in the same way as does energy-momentum conservation from co-ordinate invariance in GTR.

translations in the Minkowski space (Lorentz transformation) the conserved current can be written in the form

$$\partial_\mu T_{\mu\nu} \equiv \partial_\mu \left( L\delta_{\mu\nu} - \partial_\nu \varphi_r \frac{\partial L}{\partial \partial_\mu \varphi_r} \right) = 0$$

where $\varphi_r$ denote functions on which (and their first derivatives) the Lagrangian may depend.

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

provides the elements of $T_{\mu\nu}$ in which - according to the left side - the contribution of the kinetic and potential components are mixed by the $g_{\mu\nu}$ curvature tensor. Applying the usual integration method and Gauss' theorem, we get the fourth column of the momentum-energy stress tensor for a conserved quantity, what is no else than the four-momentum density, which behaves like a four-vector and whose individual components are

$$P_\nu = \frac{1}{ic} \int T_{4\nu} \mathrm{d}V$$

or separated

$$P_k = \frac{1}{ic} \int T_{4k} \mathrm{d}V = \frac{1}{ic} \int \partial_k \varphi_i \frac{\partial L}{\partial \partial_4 \varphi_i} \mathrm{d}V \qquad (k = 1, 2, 3)\,;$$

$$H = -icP_4 = -\int T_{44} \mathrm{d}V = \int (\partial_4 \partial_i \frac{\partial L}{\partial \partial_4 \varphi_i} - L) \mathrm{d}V$$

what are considered the conserved total momentum and energy of the field respectively.

If we take into account the qualitative difference between the masses $m_T$ (what appear in the components of $P_k$) and $m_V$ (what appears in $H$) that are mixed by the curvature tensor $g_{\mu\nu}$ in the elements of $T_{\mu\nu}$, this consideration will involve the mixed $m_T$ and $m_V$ dependence of the Lagrangians as well. As a consequence, $P_k$ and $H$ cannot be considered separately, and independently of each other, conserved quantities. (We do not investigate

here the ambiguous interpretations of invariant mass.) The covariance of the gravitational equation can no more be secured by the Lorentz invariance alone. The lost symmetry of nature can be restored only with the shown invariance between the isotopic mass states (as field charges of the gravitational field, conservation of $\Delta$) which are rotated in an isotopic field charge spin gauge field. The covariance of the gravitational equation is a result of invariance under the combination of the Lorentz transformation and rotation in the isotopic field charge field. In the latter case the four components of $(P_k[m_T], H[m_V])$ transform as isovectors. Due to the IFCS gauge transformation, the transformation of the field components can be described in a (space-time +) velocity dependent gauge field, whose metric, consequently, depends also on the velocity components, and is subject of a Finsler geometry.

# References


[1] Einstein, A. (1921) A brief outline of the development of the theory of relativity, *Nature*, 106, 2677, Feb. 17, 782-784.

[2] Einstein, A. (1935) Elementary derivation of the equivalence of mass and energy, *Bull. Amer. Math. Soc.*, 41, 223-230.

[3] von Laue, M. (1911) *Ann. Phys.* 35, 524-542.

[4] von Laue, M. (1955) *Relativitätstheorie*, 6th ed. Braunschweig.

[5] Mie, G. (1912a) *Ann. Phys.*, 39, 1-40.

[6] Mie, G. (1912b) Grundlagen einer Theorie der Materie, *Ann. Phys.*, 37, 511-534.

[7] Mie, G. (1913) *Ann. Phys.*, 40, 1-66.

[8] Hraskó, P. (2003) Ekvivalens-e egymással a tömeg és az energia? [Are mass and energy equivalent? In Hungarian] *Fizikai Szemle*, 53, 9, 330.

[9] de Broglie, L. (1923) Note. Radiations. - Ondes et quanta, *Comptes Rendus*, 177, 507-510.

[10] de Broglie, L. (1925) Recherches sur la théorie des quanta, *Annales de Physique*, 10, III, 22-128.

[11] de Haas, E. P. J. (2004a) The combination of de Broglie's harmony of the phases and Mie's theory of gravity results in a principle of equivalence for quantum gravity, *Ann. Fond. L. de Brioglie* 29, 4, 707-726.

[12] de Haas, E.P.J. (2005) From Laue's stress-energy tensor to Maxwell's Equations and the implications for Einstein's GTR, *Proceedings of the Int. Conference "hysical Interpretation of Relativity Theory (PIRT-05)"*, eds. M.C.Duffy et al., Moscow, Bauman Univ. Press.

[13] Darvas, G. (2006) Invariance, identity, equivalence, *Symmetry: Culture and Science*, 17, 175-192.

[14] Darvas, G. (2011) The Isotopic Field Charge Spin Assumption, *International Journal of Theoretical Physics*, 50, 10, 2961-2991. DOI: 10.1007/s10773-011-0796-9

[15] Darvas, G. (2012, coming) *Another Version of Facts - on Physical Interactions.*

[16] Darvas, G. (2007) A velocity dependent gauge invariance, In: *Symmetry in Nonlinear Physics*, Kiev, Institute of Mathematics, Ukrainian Academy of Sciences, http://www.imath.kiev.ua/ appmath/Abstracts2007/Darvas.html

[17] Darvas, G. (2009) Conserved Noether currents, Utiyama's theory of invariant variation, and velocity dependence in local gauge invariance, *Concepts of Physics*, VI, 1, 3-16.

[18] Hooft, Gerard't (2005) "Conceptual Basis of Quantum Field Theory" In: Elsevier *"Handbook of the Philosophy of Science"*.

[19] CERN workshop, (2008a) Report, http://arxiv.org/pdf/0801.1800

[20] CERN workshop, (2008b) Report, http://arxiv.org/pdf/0801.1833

[21] CERN workshop, (2008c) Report, http://arxiv.org/pdf/0801.1826

[22] Weinberg, S. (1997) Changing Attitudes and the Standard Model, p. 38, in Hoddeson, L., L. Brown, M. Riordan and M. Dresden: *The Rise of the Standard Model*, Cambridge University Press, Cambridge., pp. 3644.

[23] Yang, C. N., Mills, R. L. (1954) Conservation of isotopic spin and isotopic gauge invariance, *Phys. Rev.* 96 (1), 191-195.

[24] Mills, R (1989) Gauge fields, *American Journal of Physics*, 57 (6) June, 493-507.

[25] Hooft, Gerard't (2002) *Introduction to General Relativity*, Princeton: Rinton Press Inc.

[26] Martin, C.A. (2003) On continuous symmetries and the foundations of modern physics, in Brading, K. and Castellani, E., editors, *Symmetries in Physics: Philosophical Reflections*, pp. 29-60, Cambridge: Cambridge University Press, 445 p.

[27] Weyl, H. (1918) Gravitation und Elektrizität, *Sitzungsberichte Preussische Akademie der Wissenschaften*, 465-480.

[28] Weyl, H. (1929) Electron and gravitation, (Engl. transl. in: O'Raifeartaigh: *The Dawning of Gauge Theory*, 1997, Princeton Univ. Press).

[29] Higgs, P.W. (1966) Spontaneous symmetry breakdown without massless bosons, *Phys. Rev.* 145, 1156-1162.

[30] Wigner, E.P. (1984) The meaning of symmetry, pp. 729-733, in Zichichi, A. ed., *Gauge interactions theory and experiment*, Proceedings of the $20^{\text{th}}$ international school of subnuclear physics, Erice, Sicily, New York: Plenum Press.

[31] Noether, E. A. (1918) Invariante Variationsprobleme, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen: Mathematisch-physikalische Klasse*, 235-257.

[32] Itzykson, C., and Zuber, J.B. (1980) *Quantum Field Theory*, New York: McGraw-Hill.

[33] Weinberg, S. (1973) General theory of broken symmetries, *Phys. Rev.* D 7, 1068-1082

[34] Castellani, E. (2001) Reductionism, emergence and effective field theories, *arXiv:physics/0101039 v1*, 18 p.

[35] Darvas, G. (2011) GTR and the Isotopic Field Charge Spin Assumption, pp. 53-65, In: *Physical Interpretations of Relativity Theory*, Ed. M.C. Duffy, V.O. Gladyshev, A.N. Morozov, P. Rowlands, Bauman Moscow State Technical University: Moscow, Liverpool, Sunderland, 347 p.

ALGEBRAS GROUPS AND GEOMETRIES 29, 303 - 308 (2012) 

# ON THOMPSON'S CONJECTURE FOR ALTERNATING GROUP $A_{16}$

**Jianxing Bi***
School of Mathematics, Liaoning University
Shenyang, 110036, P.R. China
mabjx@sina.com

Received September 1, 2012

## Abstract

In this paper author proves that
(1) if G is a finite group with $Z(G) = 1$, $M$ is a nonabelian simple group and $N(G) = N(M)$ then $|M|$ divides $|G|$, where $N(G) = \{n \in N|\ G$ has a conjugacy class of size $n\}$;
(2) if $M = A_{16}$ in (1) then $G \cong A_{16}$.

**Keywords:** Alternating group; Sylow subgroup; Conjugacy class; Normal subgroup.

*This work was supported by the National Natural Science Foundation of China (10771172) and the Science Foundation of Liaoning Provincial Department of Education(05L153)

Copyright © 2012 by Hadronic Press Inc., Palm Harbor, FL 34682, U.S.A.

In 1980s Professor J.G.Thompson conjectured if G is a finite group with $Z(G) = 1$, $M$ is a nonabelian simple group and $N(G) = N(M)$ then $G \cong M$.

If G is a finite group let $t(G)$ denote the number of the prime graph components of $G$. For $t(M) > 1$ Professor Guiyun Chen had proved $|G| = |M|$ (See [5] and [6]). Further Guiyun Chen, A. Iranmanesh, B. Khosravi, S.H. Alavi, et al, gave positive answer to Thompson's conjecture for $t(M) > 1$. For $t(M) = 1$, it is difficult to answer to Thompson's conjecture. A.V.Vasil'ev has given positive answer to Thompson's conjecture for $M = A_{10}$ (we know $t(A_{10}) = 1$) (See [10]). In this paper author gives positive answer to Thompson's conjecture for $M = A_{16}$ (we know $t(A_{16}) = 1$).

If $G$ is a finite group then $\pi(G)$ denotes the set of the prime factors of $|G|$. If $x \in G$ let $x^G$ denote the conjugacy class of $x \in G$. If $n$ is a natural number and $r$ is a prime let $|n|_r$ denote the $r$-part of $n$. If $A$ is a finite nonempty set and $r$ is a prime let $|A|_r$ denote the $r$-part of $|A|$. We can write $\sigma$ as a product of disjoint cycles if $\sigma$ is a permutation of degree $n$. The type of $\sigma$ is the expression $type(\sigma) = 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}$ where $\lambda_i$ is the number of $i$-cycles of $\sigma(1^{\lambda_1}$ may be omitted and $i^{\lambda_i}$ may be omitted for $\lambda_i = 0)$. In the proofs we use the following facts: If $P$ is a $p-$group ($p$ prime) and $P_1 \neq 1$ is a normal subgroup of $P$ then there exists $x \in Z(P) \cap P_1$ with $o(x) = p$. (In fact $\langle x \rangle$ is a minimal normal subgroup of $P$ and $\langle x \rangle \leq P_1$).

**Lemma 1.** *Let $G$ be a finite group. If $N$ is a normal subgroup of $G$ and $x \in G$ then $|(\bar{x})^{G/N}|[N : C_N(x)]$ divides $|x^G|$, where $\bar{x} = xN$.*

*Proof.* $|x^G| = [G : C_G(x)] = [G : C_G(x)N][C_G(x)N : C_G(x)]$
$= [G/N : C_G(x)N/N][C_G(x)N : C_G(x)]$
$= [G/N : C_{G/N}(\bar{x})][C_{G/N}(\bar{x}) : C_G(x)N/N][C_G(x)N : C_G(x)]$.
Since $[C_G(x)N : C_G(x)] = [N : C_N(x)]$ we have that $|(\bar{x})^{G/N}|[N : C_N(x)]$ divides $|x^G|$.

**Lemma 2.** *Let $1 \neq x \in A_{16}$. If 13 does not divide $|x^{A_{16}}|$ then $2 \cdot 5 \cdot 7$ divides $|x^{A_{16}}|$.*

*Proof.* Since 13 does not divide $|x^{A_{16}}|$ we have that 13 divides $|C_{A_{16}}(x)|$. Thus $type(x) = 13^1$ or $type(x) = 3^1 13^1$ or $type(x) = 3^1$. Further we can prove that $2 \cdot 5 \cdot 7$ divides $|x^{A_{16}}|$.

**Lemma 3.** *Let $G$ be a finite group, $N$ a normal subgroup of $G$ and $p \in \pi(N)$. If $r \in \pi(G)$ and $r \notin \pi(N)$ then $r$ divides $|[N : C_N(x)]|_p - 1$ where $x$ is a $r$-element in $G$.*

*Proof.* Let $|N|_p = p^e$ and $|C_N(x)|_p = p^t$. Since $\langle x \rangle$ is Sylow $r$-subgroup of $\langle x \rangle N$, by Lemma 5 in [3], we have $r | p^{e-t} - 1$. Therefore $r$ divides $|[N : C_N(x)]|_p - 1$.

**Lemma 4.** *Let $G$ be a finite group and $M$ nonabelian simple group. If $Z(G) = 1$ and $N(G) = N(M)$, then $\pi(G) = \pi(M)$.*

*Proof.* See Lemma 1.1 in [5].

**Lemma 5.** *Let $H$ be a finite nonabelian simple groups. If $p \in \pi(H)$, then there exists $x \in H$ such that $|x^H|_p = |H|_p$.*

*Proof.* See Theorem in [2].

**Theorem 6.** *Let $G$ be a finite group and $M$ a finite nonabelian simple group. If $Z(G) = 1$ and $N(G) = N(M)$, then $|M|$ divides $|G|$.*

*Proof.* Let $p \in \pi(M)$. If $|M|_p = p^k$ then there exists $x \in M$ such that $|x^M|_p = p^k$. Thus there exists $g \in G$ such that $|g^G|_p = p^k$. Therefore $p^k$ divides $|G|$.

Let $G$ be a finite group with $Z(G) = 1$ and $N(G) = N(A_{16})$. Suppose $1 = N_0 < N_1 < \cdots < N_m$ is a chief series of $G$. Then there exists $i$ such that $13 \in \pi(N_i)$ and $13 \notin \pi(N_{i-1})$. Let $H = N_i$ and $N = N_{i-1}$. Then $1 \leq N < H \leq G$ is a normal series of $G$, $H/N$ is a minimal normal subgroup of $G/N$ and $13 \in \pi(H)$ and $13 \notin \pi(N)$.

**Lemma 7.** *Let $G$ be a finite group with $Z(G) = 1$ and $N(G) = N(A_{16})$. Let $G \geq H > N \geq 1$ be a normal series of $G$ and $H/N$ a minimal normal subgroup of $G/N$. If $13 \in \pi(H/N)$ and $13 \notin \pi(N)$, then $H/N$ is isomorphic to one of the nonabelian simple groups in Table 1.*

*Proof.* $H/N$ is a direct product of nonabelian simple groups whose are mutually isomorphic or is a elementary Abelian 13-group since $H/N$ a minimal normal subgroup of $G/N$. Suppose that $H/N$ is a elementary Abelian 13-group. Let $P_1 \in Syl_{13}(H)$, $P \in Syl_{13}(G)$ and $P_1 = P \bigcap H$. Then there exists $g \in P_1 \bigcap Z(P)$ with $o(g) = 13$. Thus $|g^G|_{13} = 1$. We prove $|g^G|_7 = 1$ in the following.

Since $7^2 = |A_{16}|_7 \geq |g^G|_7 \geq |[N : C_N(g)]|_7$ by Lemma 1 and 13 divides $|[N : C_N(g)]|_7 - 1$ by Lemma 3 we have $|[N : C_N(g)]|_7 = 1$. Let $a \in G$ be 7-element. We write $\bar{a} = aN$. Since $13 = |A_{16}|_{13} \geq |(\bar{a})^{G/N}|_{13} \geq |[H/N : C_{H/N}(\bar{a})]|_{13}$ by Lemma 1 and 7 divides $|[H/N : C_{H/N}(\bar{a})]|_{13} - 1$ we have $|[H/N : C_{H/N}(\bar{a})]|_{13} = 1$. Therefore $\bar{a}$ centralizes $H/N$. So $\bar{a}^{-1}\bar{g}\bar{a} = \bar{g}$ where $\bar{g} = gN$. Thus $\langle g \rangle N = \langle a^{-1}ga \rangle N$. Since $\langle g \rangle$ and $\langle a^{-1}ga \rangle$ are Sylow

13-subgroups of $\langle g\rangle N$ and $N$ is a normal subgroup of $\langle g\rangle N$ there exists $c \in N$ such that $c^{-1}\langle g\rangle c = \langle a^{-1}ga\rangle$. Thus $(ac^{-1})^{-1}\langle g\rangle(ac^{-1}) = \langle g\rangle$. Assume $(ac^{-1})^{-1}g(ac^{-1}) = g^k (1 \leq k \leq 12)$. Thus $\bar{a}^{-1}\bar{g}\bar{a} = (\bar{g})^k$. Since $\bar{a}^{-1}\bar{g}\bar{a} = \bar{g}$ we have $k = 1$. Therefore $(ac^{-1})^{-1}g(ac^{-1}) = g$. Hence $ac^{-1} \in C_G(g)$. Thus $a \in C_G(g)N$. It follows that $|[G : C_G(g)N]|_7 = 1$.

Now we have

$$|g^G|_7 = |[G : C_G(g)N]|_7|[C_G(g)N : C_G(g)]|_7 = |[N : C_N(g)]|_7 = 1.$$

From Lemma 2 we have that $|x^{A_{16}}|_{13} = 13$ or $|x^{A_{16}}|_7 = 7$ for $1 \neq x \in A_{16}$, contrary to $|g^G|_{13} = 1$ and $|g^G|_7 = 1$.

Thus $H/N$ is a direct product of nonabelian simple groups whose are mutually isomorphic. Suppose $H/N = K_1 \times K_2 \times \cdots \times K_u$,where $K_1, K_2, \cdots, K_u$ are mutually isomorphic nonabelian simple groups. Since the maximal prime factor of $|H/N|$ is 13 by Lemma 4 we have that the maximal prime factor of $|K_1|$ is 13. Assume $u \geq 2$. From Lemma 5, there exist $a \in K_1$ and $b \in K_2$ such that $|a^{K_1}|_{13} = |K_1|_{13}$ and $|b^{K_2}|_{13} = |K_2|_{13}$. By Lemma 1, $13 = |A_{16}|_{13} \geq |(ab)^{G/N}|_{13} \geq |(ab)^{H/N}|_{13} = |K_1|_{13}|K_2|_{13} = (|K_1|_{13})^2$, a contradiction. Thus $u = 1$, i.e. $H/N$ is a nonabelian simple group and $|H/N|_{13} = 13$. If $p \in \pi(H/N)$ then there exists $x \in H/N$ such that $|x^{H/N}|_p = |H/N|_p$ by Lemma 5. Hence $|H/N|_p = |x^{H/N}|_p \leq |x^{G/N}|_p \leq |A_{16}|_p$. Therefore $|H/N|$ divides $|A_{16}|$. From [8](p139-142), $H/N$ is isomorphic to one of the nonabelian simple groups in Table 1.

Table 1

| $G$ | $\lvert G\rvert$ | $\lvert Out(G)\rvert$ | $G$ | $\lvert G\rvert$ | $\lvert Out(G)\rvert$ |
|---|---|---|---|---|---|
| $L_2(13)$ | $2^2 \cdot 3 \cdot 7 \cdot 13$ | 2 | $L_3(3)$ | $2^4 \cdot 3^3 \cdot 13$ | 2 |
| $L_2(25)$ | $2^3 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $L_2(27)$ | $2^2 \cdot 3^3 \cdot 7 \cdot 13$ | 6 |
| $Sz(8)$ | $2^6 \cdot 5 \cdot 7 \cdot 13$ | 3 | $U_3(4)$ | $2^6 \cdot 3 \cdot 5^2 \cdot 13$ | 4 |
| $L_2(64)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 6 | $G_2(3)$ | $2^6 \cdot 3^6 \cdot 7 \cdot 13$ | 2 |
| $L_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 13$ | 4 | ${}^2F_4(2)'$ | $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ | 2 |
| $L_3(9)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ | 4 | ${}^3D_4(2)$ | $2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ | 3 |
| $G_2(4)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ | 2 | $S_4(8)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ | 6 |
| $A_{13}$ | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 | $A_{14}$ | $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |
| $A_{15}$ | $2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ | 2 | $A_{16}$ | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |

**Corollary 8.** *Let $G$ be a finite group with $Z(G) = 1$ and $N(G) = N(A_{16})$. If $1 \leq N < H \leq G$ is normal series of $G$, $13 \in \pi(H)$ but $13 \notin \pi(N)$, then $H/N$ is nonsolvable.*

**Theorem 9.** *Let $G$ be a finite group. If $Z(G) = 1$ and $N(G) = N(A_{16})$, then $G \cong A_{16}$.*

*Proof.* By Lemma 7 there exists normal series $1 \leq N < H \leq G$ of $G$ such that $13 \notin \pi(N)$ and $H/N$ is isomorphic to one of the nonabelian simple groups in Table 1. From Table 1 we know that 13 does not divide $|Out(H/N)|$. Thus 13 does not divide $|\bar{G}/(\bar{H} \times C_{\bar{G}}(\bar{H}))|$ where $\bar{H} = H/N$ and $\bar{G} = G/N$. We prove the Theorem in six steps.

(i) $|G|_{13} = 13$.

If $|G|_{13} > 13$ then $|C_{\bar{G}}(\bar{H})|_{13} \geq 13$. Let $\bar{P} \in Syl_{13}(C_{\bar{G}}(\bar{H}))$. If $\bar{P}$ is not subgroup of the center of $C_{\bar{G}}(\bar{H})$ then there exists $\bar{x} \in C_{\bar{G}}(\bar{H})$ such that $|(\bar{x})^{C_{\bar{G}}(\bar{H})}|_{13} \geq 13$. By Lemma 5 there exists $\bar{y} \in \bar{H}$ such that $|(\bar{y})^{\bar{H}}|_{13} = 13$. Hence $|(\bar{x}\bar{y})^{\bar{G}}|_{13} \geq |(\bar{x}\bar{y})^{\bar{H}}|_{13} \geq 13^2$ by Lemma 1. Thus there exists $g \in G$ such that $|g^G|_{13} \geq 13^2$ by Lemma 1, contrary to $|A_{16}|_{13} = 13$. If $\bar{P}$ is subgroup of the center of $C_{\bar{G}}(\bar{H})$ then $\bar{P}$ is a normal subgroup of $\bar{G}$. From Corollary 8 $\bar{P}$ is nonsolvable, a contradiction.

(ii) There exists normal series $1 \leq T < K \leq G$ of $G$ such that $K/T \cong \bar{H}$ and $G/T$ is isomorphic to a subgroup of $Aut(\bar{H})$. Since $1 \leq C_{\bar{G}}(\bar{H}) < C_{\bar{G}}(\bar{H})\bar{H} \leq \bar{G}$ is normal series of $\bar{G}$ and $C_{\bar{G}}(\bar{H})\bar{H}/C_{\bar{G}}(\bar{H}) \cong \bar{H}$ there exists $T < G$ and $K \leq G$ such that $T/N = C_{\bar{G}}(\bar{H})$ and $K/N = C_{\bar{G}}(\bar{H})\bar{H}$. Thus $1 \leq T < K \leq G$ is normal series of $G$ and $K/T \cong \bar{H}$ and $G/T$ is isomorphic to a subgroup of $Aut(\bar{H})$.

(iii) $5 \notin \pi(T)$ and $7 \notin \pi(T)$.

If $5 \in \pi(T)$ we take $x \in G$ with $o(x) = 13$. Since $5^3 = |A_{16}|_5 \geq |x^G|_5 \geq |[T : C_T(x)]|_5$ by lemma 1 and 13 divides $|[T : C_T(x)]|_5 - 1$ by Lemma 3 we have $|[T : C_T(x)]|_5 = 1$. Then there exists $Q_1 \in Syl_5(T)$ such that $x$ centralizes $Q_1$. Let $Q \in Syl_5(G)$ with $Q_1 = T \bigcap Q$. Then there exists $y \in Z(Q) \cap Q_1$ with $o(y) = 5$. Thus $gcd(|y^G|, 5 \cdot 13) = 1$, contrary to Lemma 2.

Similarly we can prove $7 \notin \pi(T)$.

(iv) $K/T \cong A_{15}$ or $K/T \cong A_{16}$.

If $K/T$ is isomorphic to one of the simple groups in Table 1 except $A_{15}$ and $A_{16}$ then $|Aut(K/T)|_5 \leq 5^2$ from Table 1. Since $G/T$ is isomorphic to a subgroup of $Aut(K/T)$ we have $|G/T|_5 \leq 5^2$. Since $5 \notin \pi(T)$ by (iii) we have $|G|_5 \leq 5^2$, contrary to Theorem 6.

(v) $2 \notin \pi(T)$, $3 \notin \pi(T)$, $11 \notin \pi(T)$ and $K \cong A_{16}$.

We take $x \in G$ with $o(x) = 13$. If $|[T : C_T(x)]|_2 \neq 1$, then $|[T : C_T(x)]|_2 \geq 2^{12}$ since $exp_{13}(2) = 12$ and 13 divides $|[T : C_T(x)]|_2 - 1$ by

Lemma 3. Since $\bar{x} = xT$ is an element of order 13 in $K/T$ and $K/T \cong A_{15}$ or $K/T \cong A_{16}$ we have that $\bar{x}$ is a 13-cycle and $|(\bar{x})^{K/T}|_2 = 2^k \cdot 3^s \cdot 7^2 \cdot 11$ where $k \geq 10$ and $s \geq 5$ Thus $2^{14} = |A_{16}|_2 \geq |x^G|_2 \geq |(\bar{x})^{K/T}|_2 |[T : C_T(x)]|_2 \geq 2^{22}$ by lemma 1, a contradiction. If $|[T : C_T(x)]|_2 = 1$ we take $P \in Syl_2(T)$. Thus $13 \in \pi(C_K(P))$. By Frattini's argument, we have $K = N_K(P)T$. Thus $K/T = N_K(P)K/T \cong N_K(P)/N_T(P)$. Therefore $N_K(P)/N_T(P)$ is isomorphic to $A_{15}$ or $A_{16}$. Thus $C_K(P)N_T(P) = N_T(P)$ or $C_K(P)N_T(P) = N_K(P)$. Since $13 \in \pi(C_K(P))$ we have $C_K(P)N_T(P) = N_K(P)$. Thus $7^2$ divides $|C_K(P)|$. Therefore $gcd(|y^G|, 7 \cdot 13) = 1$ for $1 \neq y \in P$, contrary to Lemma 2.

Similarly we can prove that $3 \notin \pi(T)$ and $11 \notin \pi(T)$.

Now we have $T = 1$. Thus $K \cong A_{15}$ or $K \cong A_{16}$. By Theorem 6 and (ii), $|A_{16}| \leq |G| \leq |Aut(K)|$, so $K \cong A_{16}$

(vi) $G \cong A_{16}$.

From (ii) and (v) $G \cong S_{16}$ or $G \cong A_{16}$. If $G \cong S_{16}$ we take a 2-cycle $x \in S_{16}$. We have $|x^{S_{16}}| = 2^3 \cdot 3 \cdot 5$. Thus $gcd(|x^G|, 7 \cdot 11) = 1$ contrary to Lemma 2.

# References


[1] S.H.Alavi, A.Daneshkhah, A new characterization of alternating and symmetric groups, J. Appl. Math. Computing 17(2005), 245-258.

[2] Jianxing Bi, An arithmetical property of the conjugacy classes of the finite simple groups, J. Liaoning University(Natural Sciences Edition), 35(1)(2008), 7-8.(Chinese)

[3] Jianxing Bi, A Characterization of $L_n(q)$ by the Normalizer's orders of Their Sylow Subgroups, Acta. Math. Sinica, New series, 11(3)(1995), 300-306.

[4] G.Y.Chen, Further reflections on Thompson's conjecture, J.Algebra,218(1999),276.

[5] G.Y. Chen, On Thompson's conjecture, J. Algebra, 185(1996), 184 193.

[6] G.Y. Chen, On Thompson's conjecture, Ph. D. Th., Sichuan Univ., Chengdu, 1994.

[7] J. H. Conway, et al, ATLAS of Finite Groups, Clarendon Press, Oxford, 1985.

[8] A. Iranmanesh, S.H. Alavi and B. Khosravi, A characterization of PSL(3,q) where q is an odd prime power, J. Pure and Appl. Algebra, 170(2002), 243-254.

[9] B. Khosravi, et al, A new characterization of $PSU(p, q)$, Acta Math. Hungar., 107(3)(2005), 235-252.

[10] A.V.Vasil'ev, On Thompson's Conjecture, Siberian Electr.Math.Rep., 6(2009),457.

ALGEBRAS GROUPS AND GEOMETRIES 29, 309 - 376 (2012) 

# ALGEBRA-INVARIANT MODELS FOR NONLINEAR NONLOCAL MEDIA[1]

**Danylenko V.A., Danevich T.B., Makarenko O.S.[2], Ukraine, Moskaliuk N.M.[3], Skurativskiy S.I. and Vladimirov V.A.**
Subbotin Institute of Geophysics of the National Academy of Sciences of Ukraine
Palladin Av. 32, Kyiv, Ukraine, UA-03680
makalex@i.com.ua

Received October 3, 2012

## Abstract

In this article, mathematical models of nonequilibrium media are derived within the algebra-invariant approach. Using the methods of thermodynamics of irreversible processes and general assumptions concerning the kinetics of internal variables, the generalized nonlocal dynamical equations of state are constructed and their connections with well-known models are considered. Using monographs [217, 218, 219, 220, 221], the group-theoretical classification of generalized models and their invariant solutions will be presented in our next articles in more detail.

---

[1]Based on invited talks given by A. Makarenko at the 5th Petrov International Symposium on *High Energy Physics, Cosmology and Gravity*, Kyiv (Ukraine), April 29–June 15, 2012.

[2]On leave from the National Technical University of Ukraine (KPI)

[3]On leave from Bogolyubov Institute for Theoretical Physics, Ukraine.

Copyright © 2012 by Hadronic Press Inc., Palm Harbor, FL 34682, U.S.A.

# 1 Phenomenological relations in the vicinity of local equilibrium

It is known that linear thermodynamics of irreversible processes is based on the conjecture about linear connections between generalized fluxes and forces. In nonlinear nonequilibrium thermodynamics, constitutive relations should take into account terms higher than the first order or dependence of the phenomenological coefficients on thermodynamical forces.

According to Lykov's work [181], in unsteady intensive processes, the generalized thermodynamical fluxes $J_i$ are connected with the generalized thermodynamical forces $X_k$ by means of some nonlinear relations, form of which are unknown.

However, if the displacement from equilibrium is not large, the nonlinear dependence between fluxes and forces can be described approximately by the linear relations of the following form:

$$J_i = L_i^r \dot{J}_i + \sum_k \left( L_{ik} X_k + L'_{ik} \dot{X}_k \right), \tag{1}$$

where $L_i^r, L_{ik}, L'_{ik}$ are the constant phenomenological coefficients.

In the case of strongly nonequilibrium processes, the flow gradients are large and real media manifest effects of their internal structure. Under these conditions, expressions for the thermodynamical fluxes are nonlocal in both space and time. The phenomenological relations can be written as form [65]:

$$J_i = \sum_j \int_r d\vec{r} \int_{-\infty}^{t} dt' K_{ij} \left( \vec{r}, \vec{r}', t, t'; \{ l_\varepsilon, \tau \} \right) X_j \left( \vec{r}', t' \right),$$

where $K_{ij}$ is the kernel; $\{ l_\varepsilon, \tau \}$ are the set of spatial and temporal scales of internal medium structure. The integral equation of state has the following form [65]:

$$p\left(x,t\right) = \int_{-\infty}^{t} \frac{dt'}{\varepsilon_t} \int_{-\infty}^{\infty} \frac{dx'}{\varepsilon_r} L\left(x, x', t, t'; \varepsilon_r, \varepsilon_t\right) p_0\left(x', t'\right),$$

where $\varepsilon_t = \dfrac{\tau}{\theta}$, $\theta$ is the characteristic time of a process; $\tau$ is the time of relaxation, $\varepsilon_r = \dfrac{l_\varepsilon}{l}$ ($l_\varepsilon$ is the characteristic scale of internal medium structure; $l$ is the characteristic scale of a process).

For strongly nonequilibrium processes in case of small parameters of spatial and temporal nonlocalities ($\varepsilon_r, \varepsilon_t << 1$ ) the effects of spatial and temporal nonlocalities can be separated and it is possible to pass from an integral model to a weakly nonlocal one [65]:

$$p(x,t) = \left( \sum_{n=0}^{\infty} \frac{\tilde{\varphi}_n''(t,\varepsilon_t)}{n!} \frac{\partial^n}{\partial t^n} \sum_{k=0}^{\infty} \frac{\tilde{\varphi}_k'(x,\varepsilon_r)}{k!} \frac{\partial^k}{\partial x^k} \right) p_0(x,t),$$

where $p$ is the pressure; $\tilde{\varphi}_n''(t,\varepsilon_t), \tilde{\varphi}_k'(x,\varepsilon_r)$ are the functions describing a medium structure. Depending on the choice of $n, k$ and the values of the coefficients, we can obtain different models. In particular, at $n = \overline{0,2}$, $k = \overline{0,2}$ and zero some coefficients we have the equation with operator in the form:

$$\hat{L} = \left(1 + \tau \frac{d}{dt} - \tau^2 \frac{d^2}{dt^2}\right) + \varepsilon_r \left(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right).$$

In this case it is possible to obtain the following constitutive relation in the vicinity of local equilibrium [45], [46]:

$$\frac{d\xi}{dt} = \Gamma \left[ A + \varepsilon_r \left( \xi_{xx} - \frac{\upsilon_x}{\upsilon} \xi_x \right) \right] + \tau_{\text{TP}} \frac{d^2\xi}{dt^2}, \tag{2}$$

where $\xi$ is the internal variable; $A$ is the affinity of the relaxing process in media with temporal nonlocality; $\upsilon$ is the specific volume; $A + \varepsilon_r \left( \xi_{xx} - \dfrac{\upsilon_x}{\upsilon} \xi_x \right)$ is the affinity of the relaxing process in media with spatial and temporal nonlocalities; $\Gamma$ is the kinetic parameter; $\tau_{\text{TP}}$ is the time of relaxation (the parameter of temporal nonlocality); $\varepsilon_r$ is the parameter of spatial nonlocality, $\varepsilon_r \equiv l_\varepsilon^2 \left( \dfrac{\partial^2 E}{\partial \xi^2} \right)_0$ [43] ; $E$ is specific internal energy; the second time derivative takes into account the nonstationarity of the generalized thermodynamical fluxes.

The general phenomenological relation taking into account the dynamics of structure elements has the following form:

$$\sum_{n=0}^{\mathrm{N}} (-1)^n \tau_{\mathrm{TP}}^n \frac{d^{(n+1)}}{dt^{(n+1)}} \xi = \Gamma \sum_{k=0}^{\mathrm{K}} (-1)^k \tau_{\mathrm{TP}}^k \frac{d^k}{dt^k} \left[ \mathrm{A} + \varepsilon_r \left( \frac{\upsilon_x}{\upsilon} \right) \sum_{m=0}^{\mathrm{M}} (-1)^{(m+1)} \left( \frac{\upsilon}{\upsilon_x} \right)^m \frac{d^{(m+1)}}{dx^{(m+1)}} \xi \right] \tag{3}$$

At M = 1 from relation (3) it follows the phenomenological relation offered in paper [49] for the construction of mathematical models of nonlinear nonlocal media with internal oscillators:

$$\sum_{n=0}^{\mathrm{N}} (-1)^n \tau_{\mathrm{TP}}^n \frac{d^{(n+1)}}{dt^{(n+1)}} \xi = \Gamma \sum_{k=0}^{\mathrm{K}} (-1)^k \tau_{\mathrm{TP}}^k \frac{d^k}{dt^k} \left[ \mathrm{A} + \varepsilon_r \left( \xi_{xx} - \frac{\upsilon_x}{\upsilon} \xi_x \right) \right]. \tag{4}$$

If N = 0, K = 0, M = 1, $\varepsilon_r \neq 0$, then from relation (3) spatially nonlocal generalization of the Onsager's relation follows (the case of steady generalized fluxes and forces) [43]:

$$\frac{d\xi}{dt} = \Gamma \left[ A + \varepsilon_r \left( \xi_{xx} - \frac{\upsilon_x}{\upsilon} \xi_x \right) \right].$$

Constitutive relation (1) is the partial case of relation (3) at N = 1, K = 1, $\varepsilon_r = 0$ offered by Lykov [181] for the description of highly intensive unsteady processes of transfer nearby equilibrium.

At $\mathrm{N} = 1, \mathrm{K} = 1, \mathrm{M} = 1, \varepsilon_{\mathrm{r}} \neq 0$ from (3) it follows:

$$\frac{d\xi}{dt} = \Gamma \left[ A + \varepsilon_r \left( \xi_{xx} - \frac{\upsilon_x}{\upsilon} \xi_x \right) \right] - \Gamma \tau_{\mathrm{TP}} \frac{d}{dt} \left[ A + \varepsilon_r \left( \xi_{xx} - \frac{\upsilon_x}{\upsilon} \xi_x \right) \right] + \tau_{\mathrm{TP}} \frac{d^2 \xi}{dt^2}. \tag{5}$$

At N = 2, K = 1, M = 1, $\varepsilon_r \neq 0$ from relation (3) the phenomenological relation [48] follows:

$$\frac{d\xi}{dt} = \Gamma \left[ A + \varepsilon_r \left( \xi_{xx} - \frac{\upsilon_x}{\upsilon} \xi_x \right) \right] - \Gamma \tau_{\mathrm{TP}} \frac{d}{dt} \left[ A + \varepsilon_r \left( \xi_{xx} - \frac{\upsilon_x}{\upsilon} \xi_x \right) \right] + \tau_{\mathrm{TP}} \frac{d^2 \xi}{dt^2} - \tau_{\mathrm{TP}}^2 \frac{d^3 \xi}{dt^3}. \tag{6}$$

At N = 2, K = 0, M = 1, $\varepsilon_r \neq 0$ relation (3) takes on the form [47]:

$$\frac{d\xi}{dt} = \Gamma\left[A + \varepsilon_r\left(\xi_{xx} - \frac{\upsilon_x}{\upsilon\xi_x}\right)\right] + \tau_{\mathrm{TP}}\frac{d^2\xi}{dt^2} - \tau_{\mathrm{TP}}^2\frac{d^3\xi}{dt^3}. \tag{7}$$

If N = 2, K = 0, M = 2, $\varepsilon_r \neq 0$, then we have:

$$\frac{d\xi}{dt} = \Gamma\left[A + \varepsilon_r\left(-\frac{\upsilon}{\upsilon_x}\xi_{xxx} + \xi_{xx} - \frac{\upsilon_x}{\upsilon}\xi_x\right)\right] + \tau_{\mathrm{TP}}\frac{d^2\xi}{dt^2} - \tau_{\mathrm{TP}}^2\frac{d^3\xi}{dt^3}. \tag{8}$$

Neglecting the spatial nonlocality ($\varepsilon_r = 0$) from relation (3) the generalized phenomenological relation for media with temporal nonlocality can be obtained

$$\sum_{n=0}^{\mathrm{N}-1}(-1)^n\tau_{\mathrm{TP}}^n\frac{d^{(n+1)}}{dt^{(n+1)}}\xi = \Gamma\sum_{k=0}^{\mathrm{K}}(-1)^k\tau_{\mathrm{TP}}^k\frac{d^k}{dt^k}\mathrm{A}. \tag{9}$$

If in the last relation we put K = 0, then we obtain the generalized constitutive relation of the N-th order for relaxing media with temporal nonlocality for the case of steady thermodynamical forces:

$$\sum_{n=0}^{\mathrm{N}-1}(-1)^n\tau_{\mathrm{TP}}^n\frac{d^{(n+1)}}{dt^{(n+1)}}\xi = \Gamma\mathrm{A}. \tag{10}$$

In case, when N = 4, we have the phenomenological constitutive relation of the 4-th order

$$\frac{d\xi}{dt} = \Gamma\mathrm{A} + \tau_{\mathrm{TP}}\frac{d^2\xi}{dt^2} - \tau_{\mathrm{TP}}^2\frac{d^3\xi}{dt^3} + \tau_{\mathrm{TP}}^3\frac{d^4\xi}{dt^4}. \tag{11}$$

Let us note that the relations between fluxes and forces must satisfy the second law of thermodynamics $\dfrac{dS}{dt} = \sum\limits_{i,k} J_i X_k \geq 0$ and the thermodynamical criterion of stability $\delta E = 0, \delta^2 E > 0$, where $E = E(\upsilon, S)$ is the specific internal energy, $S$ is the specific entropy; $\delta$ is the variation.

# 2 Dynamical equations of state for media with temporal and spatial nonlocalities with the second time and space derivatives

Let us construct the dynamical equation of state taking into account the spatial, temporal nonlocalities, and physical nonlinearity in media with internal variables near equilibrium. Some results of the construction of the dynamical equations of state with internal variables at linear relations between fluxes and forces without taking into account spatial nonlocality are shown in papers [19, 26, 65, 225, 248, 249, 260, 261].

Take the equation of state for a thermodynamical system in the following general form:

$$A = A\left(p, T, \xi\right), \quad \upsilon = \upsilon\left(p, T, \xi\right).$$

Suppose that the initial state of equilibrium is uniform:

$$S \equiv S_0, \rho \equiv \rho_0, \xi \equiv \xi_0\left(\rho_0, S_0\right), \upsilon \equiv \upsilon_0, T \equiv T_0,$$

where $\rho$ is the density, $T$ is the temperature.

Expanding function $A$ and $\upsilon$ in the Taylor series in the vicinity of equilibrium (we are confined to terms of the first order only):

$$A\left(p, T, \xi\right) = \left(\frac{\partial A}{\partial p}\right)_{\mathrm{T}\xi}\left(p - p_0\right) + \left(\frac{\partial A}{\partial T}\right)_{\mathrm{P}\xi}\left(T - T_0\right) + \left(\frac{\partial A}{\partial \xi}\right)_{\mathrm{PT}}\left(\xi - \xi_0\right), \tag{12}$$

$$\upsilon\left(p, T, \xi\right) - \upsilon_0\left(p_0, T_0, \xi_0\right) = \left(\frac{\partial \upsilon}{\partial p}\right)_{\mathrm{T}\xi}\left(p - p_0\right) + \left(\frac{\partial \upsilon}{\partial T}\right)_{\mathrm{P}\xi}\left(T - T_0\right) + \left(\frac{\partial \upsilon}{\partial \xi}\right)_{\mathrm{PT}}\left(\xi - \xi_0\right). \tag{13}$$

Suppose that constitutive relation (2) is valid near the local equilibrium.

$$\frac{d\xi}{dt} = \Gamma\left[A + \varepsilon_r\left(\xi_{xx} - \frac{\upsilon_x}{\upsilon}\xi_x\right)\right] + \tau_{\mathrm{TP}}\frac{d^2\xi}{dt^2}.$$

According to [71]

$$\left(\frac{\partial A}{\partial \xi}\right)_{\mathrm{TP}} \equiv -\frac{1}{\Gamma\tau_{\mathrm{TP}}} \Rightarrow \Gamma = -\frac{1}{\tau_{\mathrm{TP}}}\frac{1}{\left(\frac{\partial A}{\partial \xi}\right)_{\mathrm{TP}}},$$

then using (12) we transform (2):

$$\frac{d\xi}{dt} = -\frac{1}{\tau_{\rm TP}}\left\{\frac{(\partial A/\partial p)_{\xi \rm T}}{(\partial A/\partial \xi)_{\rm TP}}(p-p_0) + \frac{(\partial A/\partial T)_{\xi \rm P}}{(\partial A/\partial \xi)_{\rm TP}}(T-T_0) + (\xi-\xi_0)\right\} + \\ +\tau_{\rm TP}\frac{d^2\xi}{dt^2} + \Gamma\varepsilon_r\left(\xi_{xx} + \rho^{-1}\xi_x\rho_x\right). \tag{14}$$

Using the well-known thermodynamical relations [15]:

$$\left(\frac{\partial A}{\partial p}\right)_{\xi \rm T} = -\left(\frac{\partial A}{\partial \xi}\right)_{\rm TP}\left(\frac{\partial \xi}{\partial p}\right)_{\rm AT}, \left(\frac{\partial A}{\partial T}\right)_{\xi \rm P} = -\left(\frac{\partial A}{\partial \xi}\right)_{\rm TP}\left(\frac{\partial \xi}{\partial T}\right)_{\rm AP},$$

we present (14) in the form:

$$\frac{d\xi}{dt} = -\frac{1}{\tau_{\rm TP}}\left\{(\xi-\xi_0) - \left(\frac{\partial \xi}{\partial p}\right)_{\rm AT}(p-p_0) - \left(\frac{\partial \xi}{\partial T}\right)_{\rm AP}(T-T_0)\right\} + \\ +\tau_{\rm TP}\frac{d^2\xi}{dt^2} + \Gamma\varepsilon_r\left(\xi_{xx} + \rho^{-1}\xi_x\rho_x\right). \tag{15}$$

Apply the operator

$$1 + \tau_{\rm TP}\frac{d}{dt} - \tau_{\rm TP}^2\frac{d^2}{dt^2} \equiv \hat{L}$$

for equation (13):

$$\hat{L}(\upsilon-\upsilon_0) = \hat{L}\left(\frac{\partial \upsilon}{\partial p}\right)_{\xi \rm T}(p-p_0) + \hat{L}\left(\frac{\partial \upsilon}{\partial T}\right)_{\xi \rm P}(T-T_0) + \hat{L}\left(\frac{\partial \upsilon}{\partial \xi}\right)_{\rm TP}(\xi-\xi_0). \tag{16}$$

From equation (15) it follows:

$$\hat{L}(\xi-\xi_0) = \left(\frac{\partial \xi}{\partial p}\right)_{\rm AT}(p-p_0) + \left(\frac{\partial \xi}{\partial T}\right)_{\rm AP}(T-T_0) + \Gamma\varepsilon_r\tau_{\rm TP}\left(\xi_{xx} + \rho^{-1}\xi_x\rho_x\right). \tag{17}$$

Substitute (17) into (16):

$$\hat{L}(\upsilon-\upsilon_0) = \left[\left(\frac{\partial \upsilon}{\partial \xi}\right)_{\rm TP}\left(\frac{\partial \xi}{\partial p}\right)_{\rm AT} + \left(\frac{\partial \upsilon}{\partial p}\right)_{\xi \rm T}\hat{L}\right](p-p_0) + \\ + \left[\left(\frac{\partial \upsilon}{\partial \xi}\right)_{\rm TP}\left(\frac{\partial \xi}{\partial T}\right)_{\rm AP} + \left(\frac{\partial \upsilon}{\partial T}\right)_{\xi \rm P}\hat{L}\right](T-T_0) + \Gamma\varepsilon_r\tau_{pT}\left(\frac{\partial \upsilon}{\partial \xi}\right)_{\rm TP}\left(\xi_{xx} + \rho^{-1}\xi_x\rho_x\right). \tag{18}$$

From equation (12) it follows:

$$(\xi-\xi_0)=\left(\frac{\partial\upsilon}{\partial\xi}\right)_{\mathrm{TP}}^{-1}\left[(\upsilon-\upsilon_0)-\left(\frac{\partial\upsilon}{\partial p}\right)_{\xi\mathrm{T}}(p-p_0)-\left(\frac{\partial\upsilon}{\partial T}\right)_{\xi\mathrm{P}}(T-T_0)\right],\Rightarrow \tag{19}$$

$$\xi_{xx}=\left(\frac{\partial\upsilon}{\partial\xi}\right)_{\mathrm{TP}}^{-1}\left[\upsilon_{xx}-\left(\frac{\partial\upsilon}{\partial p}\right)_{\xi\mathrm{T}}p_{xx}-\left(\frac{\partial\upsilon}{\partial T}\right)_{\xi\mathrm{P}}T_{xx}\right], \tag{20}$$

$$\rho^{-1}\xi_x\rho_x=\left(\frac{\partial\upsilon}{\partial\xi}\right)_{\mathrm{TP}}^{-1}\left[\rho^{-1}\rho_x\upsilon_x-\left(\frac{\partial\upsilon}{\partial p}\right)_{\xi\mathrm{T}}\rho^{-1}\rho_x p_x-\left(\frac{\partial\upsilon}{\partial T}\right)_{\xi\mathrm{P}}\rho^{-1}\rho_x T_x\right]. \tag{21}$$

Let us apply the well-known thermodynamical relations [15]:

$$\left(\frac{\partial\upsilon}{\partial\xi}\right)_{\mathrm{TP}}\left(\frac{\partial\xi}{\partial p}\right)_{\mathrm{AT}}=\left(\frac{\partial\upsilon}{\partial p}\right)_{\mathrm{AT}}-\left(\frac{\partial\upsilon}{\partial p}\right)_{\xi\mathrm{T}}, \tag{22}$$

$$\left(\frac{\partial\upsilon}{\partial\xi}\right)_{\mathrm{TP}}\left(\frac{\partial\xi}{\partial T}\right)_{\mathrm{AP}}=\left(\frac{\partial\upsilon}{\partial T}\right)_{\mathrm{AP}}-\left(\frac{\partial\upsilon}{\partial T}\right)_{\xi\mathrm{P}}. \tag{23}$$

Now we substitute (20) - (23) into (18):

$$\begin{aligned}\hat{L}\,(\upsilon-\upsilon_0)=&\left[\left(\frac{\partial\upsilon}{\partial p}\right)_{\mathrm{AT}}+\left(\frac{\partial\upsilon}{\partial p}\right)_{\xi\mathrm{T}}\left(\tau_{\mathrm{TP}}\frac{d}{dt}-\tau_{\mathrm{TP}}^2\frac{d^2}{dt^2}\right)\right](p-p_0)+\\&+\left[\left(\frac{\partial\upsilon}{\partial T}\right)_{\mathrm{AP}}+\left(\frac{\partial\upsilon}{\partial T}\right)_{\xi\mathrm{P}}\left(\tau_{\mathrm{TP}}\frac{d}{dt}-\tau_{\mathrm{TP}}^2\frac{d^2}{dt^2}\right)\right](T-T_0)+\Gamma\varepsilon_r\tau_{\mathrm{TP}}\times\\&\left[\left(\upsilon_{xx}+\frac{\rho_x\upsilon_x}{\rho}\right)-\left(\frac{\partial\upsilon}{\partial p}\right)_{\xi\mathrm{T}}\left(p_{xx}+\frac{\rho_x p_x}{\rho}\right)-\left(\frac{\partial\upsilon}{\partial T}\right)_{\xi\mathrm{P}}\left(T_{xx}+\frac{\rho_x T_x}{\rho}\right)\right].\end{aligned} \tag{24}$$

Using the definitions of equilibrium and frozen isothermal coefficients of compression and coefficients of thermal expansion [15, 165]:

$$\left(\frac{\partial\upsilon}{\partial p}\right)_{\mathrm{AT}}\equiv-\upsilon_0\chi_{\mathrm{T0}},\left(\frac{\partial\upsilon}{\partial p}\right)_{\xi\mathrm{T}}\equiv-\upsilon_0\chi_{\mathrm{T}\infty},\left(\frac{\partial\upsilon}{\partial T}\right)_{\mathrm{AP}}\equiv\upsilon_0\alpha_0,\left(\frac{\partial\upsilon}{\partial T}\right)_{\xi\mathrm{P}}\equiv\upsilon_0\alpha_\infty,$$

where $\chi_{T0}$, $\chi_{T\infty}$ are the isothermal coefficients of compression; $\alpha_0, \alpha_\infty$ are the coefficients of thermal expansion, and the well-known relations between times of relaxation [165]:

$$\tau_{TP} = \tau_{TV}\frac{\chi_{T0}}{\chi_{T\infty}} = \tau_{PV}\frac{\alpha_0}{\alpha_\infty} \Rightarrow$$

$$\frac{\chi_{T\infty}}{\chi_{T0}}\tau_{TP}\tau_{TV} = \tau_{TV}^2, \quad \frac{\tau_{TP}\tau_{TV}}{\upsilon_0\chi_{T0}} = \frac{\tau_{TV}^2}{\upsilon_0\chi_{T\infty}},$$
$$\frac{\chi_{T\infty}}{\chi_{T0}}\tau_{TP} = \tau_{TV}, \quad \frac{\tau_{TP}}{\upsilon_0\chi_{T0}} = \frac{\tau_{TV}}{\upsilon_0\chi_{T\infty}}$$

we write (24) in the form:

$$p - p_0 = \left[\frac{1}{\chi_{T0}}\left(1 - \frac{\upsilon}{\upsilon_0}\right) + \frac{\alpha_0}{\chi_{T0}}(T - T_0)\right] - \tau_{TV}\frac{dp}{dt} - \frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt} +$$

$$+\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\frac{dT}{dt} + \frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2} + \frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2} - \frac{\alpha_0^2}{\alpha_\infty\chi_{T0}}\tau_{PV}^2\frac{d^2T}{dt^2} + \Gamma\varepsilon_r \times \quad (25)$$

$$\times\left[\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xx} + \rho^{-1}\rho_x\upsilon_x\right) + \tau_{TV}\left(p_{xx} + \rho^{-1}\rho_x p_x\right) - \frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(T_{xx} + \rho^{-1}\rho_x T_x\right)\right].$$

# 3 Limiting cases: quasistatic and instant processes

If a state of considered media changes slowly, then we can assume that $0 < \tau\omega \ll 1$. In this case we can omit all terms containing the differential operator $d/dt$ in the dynamical equation of state (25). Then, the obtained dynamical equation of state is transformed into the static equation of state under equilibrium conditions when all relaxing processes have finished. If a state of media changes rapidly ($\tau\omega \Rightarrow \infty$), then the terms with the differential operator $d/dt$ (relaxing processes do not begin and a medium state is frozen) play the leading role. Let us consider the possible limiting cases in detail.

1. If $\tau_{\mathrm{TV}}\omega \Rightarrow 0, \tau_{\mathrm{TP}}\omega \Rightarrow 0, \tau_{\mathrm{PV}}\omega \Rightarrow 0; l_\varepsilon/l << 1$ ($\omega$ is the frequency), then equation of state (25) has the form:

$$p - p_0 = \left[\frac{1}{\chi_{\mathrm{T}0}}\left(1 - \frac{\upsilon}{\upsilon_0}\right) + \frac{\alpha_0}{\chi_{\mathrm{T}0}}(T - T_0)\right]. \tag{26}$$

2. If $\tau_{\mathrm{TV}}\omega \Rightarrow \infty, \tau_{\mathrm{TP}}\omega \Rightarrow \infty, \tau_{\mathrm{PV}}\omega \Rightarrow \infty; l_\varepsilon/l \Rightarrow 0$, then equation of state (25) has the form:

$$\tau_{\mathrm{TV}}\left[\frac{dp}{dt} + \frac{1}{\upsilon_0\chi_{\mathrm{T}0}}\frac{d\upsilon}{dt} - \frac{\alpha_\infty}{\chi_{\mathrm{T}\infty}}\frac{dT}{dt}\right] = \tau_{\mathrm{TV}}^2\left[\frac{d^2p}{dt^2} + \frac{1}{\upsilon_0\chi_{\mathrm{T}\infty}}\frac{d^2\upsilon}{dt^2} - \frac{\alpha_\infty}{\chi_{\mathrm{T}\infty}}\frac{d^2T}{dt^2}\right].$$

From the last expression after proceeding to limit ($\tau_{\mathrm{TV}}\omega \Rightarrow \infty$), we get the following equation of state:

$$p - p_0 = \left[\frac{1}{\chi_{\mathrm{T}\infty}}\left(1 - \frac{\upsilon}{\upsilon_0}\right) + \frac{\alpha_\infty}{\chi_{\mathrm{T}\infty}}(T - T_0)\right]. \tag{27}$$

# 4 Accounting of physical nonlinearity

Now, let us return to equation of state (25). In this equation the physical nonlinearity appears after changing the variables from $(p, \upsilon, T)$ to $(p, \upsilon, S)$.

This can be shown in the example of limiting case 1 (the long wave approximation in the absence of spatial nonlocality), when the equation of state has the form (26). It is noted above that this equation is a static equation under equilibrium conditions. It is valid when all relaxing processes have finished and the state of a medium changes slowly.

Let us pass to the equation of state $E = E(\upsilon, T)$ and write the total differential of internal energy $E$

$$dE = \left(\frac{\partial E}{\partial \upsilon}\right)_{\mathrm{T}} d\upsilon + \left(\frac{\partial E}{\partial T}\right)_{\mathrm{V}} dT,$$

where

$$\left(\frac{\partial E}{\partial \upsilon}\right)_{\mathrm{T}} = T\left(\frac{\partial p}{\partial T}\right)_{\mathrm{V}} - p, \qquad \left(\frac{\partial E}{\partial T}\right)_{\mathrm{V}} \equiv C_{\mathrm{V}},$$

$C_V$ is the specific thermal capacity at constant volume.

On the other hand the total differential of internal energy $dE$ is defined by the basic law of thermodynamics

$$dE = TdS - pdv.$$

Thus, we get the relation:

$$dE = C_{\mathrm{V}}dT + \left[T\left(\frac{\partial p}{\partial T}\right)_{\mathrm{V}} dv - pdv\right] = TdS - pdv \Rightarrow$$

$$dS = C_{\mathrm{V}}\frac{dT}{T} + \left(\frac{\partial p}{\partial T}\right)_{\mathrm{V}} dv. \tag{28}$$

From equation (26) it follows

$$\left(\frac{\partial p}{\partial T}\right)_{\mathrm{V}} = \frac{\alpha_0}{\chi_{\mathrm{T0}}}. \tag{29}$$

Let

$$C_V(T) = C_{V0} = \text{const},$$

then near the local equilibrium, the solution of equation (28) has the form:

$$S - S_0 = C_{\mathrm{V0}} \ln \frac{T}{T_0} + \int_{v_0}^{v} \frac{\alpha_0}{\chi_{\mathrm{T0}}} dv. \tag{30}$$

To evaluate the integral in the last expression we use the definition of the equilibrium isometric Gruneisen's coefficient $\Gamma_{\mathrm{V0}}$ and the fact that $v/v_0 \to 1$ near equilibrium. Thus:

$$\int_{v_0}^{v} \frac{\alpha_0}{\chi_{\mathrm{T0}}} dv = \int_{v_0}^{v} \frac{\Gamma_{\mathrm{V0}} C_{\mathrm{V0}}}{v} dv = \Gamma_{\mathrm{V0}} C_{\mathrm{V0}} \ln \frac{v}{v_0}. \tag{31}$$

Taking into account expression (31), solution (30) can be presented in the form of

$$\frac{S - S_0}{C_{\mathrm{V0}}} = \ln \left\{ \frac{T}{T_0} \left(\frac{v}{v_0}\right)^{\Gamma_{\mathrm{V0}}} \right\}.$$

It follows

$$T - T_0 = T_0 \left[ e^{\frac{S-S_0}{C_{V0}}} \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{V0}} - 1 \right]. \tag{32}$$

Using (32), we change variables from $(\upsilon, T)$ to $(\upsilon, S)$ in equation (26):

$$p - p_0 = \frac{1}{\chi_{T0}} \left\{ \left( 1 - \frac{\upsilon}{\upsilon_0} \right) + \alpha_0 T_0 \left[ e^{\frac{S-S_0}{C_{V0}}} \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{V0}} - 1 \right] \right\}. \tag{33}$$

Since $\chi_{T0} = \dfrac{\upsilon_0}{c_{T0}^2} = \dfrac{\gamma_0 \upsilon_0}{c_{S0}^2}$ ($c_{T0}$, $c_{S0}$ are the isothermal and adiabatic equilibrium velocities of sound; $\gamma_0$ is the equilibrium polytropic index), then:

$$p - p_0 = \frac{c_{S0}^2}{\gamma_0 \upsilon_0} \left( 1 - \frac{\upsilon}{\upsilon_0} \right) + \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0 \upsilon_0} \left[ \sigma(S) \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{V0}} - 1 \right]. \tag{34}$$

Equation (34) is said to be the nonlinear quasistatic equation of state taking into account both the thermal and the elastic part of internal energy. Consider the case, when, in equation (34), the elastic part is negligibly small in comparison with the thermal part:

$$1 - \frac{\upsilon}{\upsilon_0} << d_0 T_0 \left[ e^{\frac{S-S_0}{C_{V0}}} \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{V0}} - 1 \right]. \tag{35}$$

Under this condition from equation of state (34) we can obtain the following nonlinear equation:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0 \upsilon_0} \left[ \sigma(S) \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{V0}} - 1 \right], \sigma(S) = \exp\left( \frac{S - S_0}{C_{V0}} \right). \tag{36}$$

For adiabatic processes ($\sigma(S) = 1$, $S = S_0$) the equation of state has the form:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0 \upsilon_0} \left[ \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{V0}} - 1 \right]. \tag{37}$$

Equation (37) is a Tate equation.

Thus, we have shown that the dynamical equation of state for media with spatial and temporal nonlocalities reduces to the Tate equation under conditions $\tau_{\mathrm{TV}}\omega \Rightarrow 0, \tau_{\mathrm{TP}}\omega \Rightarrow 0, \tau_{\mathrm{PV}} \Rightarrow 0; l_\varepsilon/l << 1$ , and (35). When the equation of state reduces to equation (37), the sound spreads with the velocity $c_{S0}$, which is the equilibrium adiabatic velocity of sound.

According to limiting case 2 in the shortwave (highfrequency) approximation, we obtain equation of state (27). This equation is similar to equation (26) for the steady case. Thus, it is shown that under rapid changing of a state, the equation of state has the form of a static equation of state with frozen relaxing process.

Now let us pass from variables $(p, v, T)$ to variables $(p, v, S)$ in equation (27) . From the basic equation of thermodynamics of irreversible processes

$$dE = TdS - pdv - Ad\xi,$$

where

$$\left(\frac{\partial E}{\partial v}\right)_{\mathrm{T}\xi} = T\left(\frac{\partial p}{\partial T}\right)_{\mathrm{V}\xi} - p,$$

$$C_{\mathrm{V}\infty} = \left(\frac{\partial E}{\partial T}\right)_{\mathrm{V}\xi},$$

$C_{\mathrm{V}\infty}$ is the frozen specific thermal capacity at constant volume, it follows:

$$dS = C_{\mathrm{V}\infty}\frac{dT}{T} + \left(\frac{\partial p}{\partial T}\right)_{\mathrm{V}\xi} dv. \tag{38}$$

Using the definition of the frozen isometric Gruneisen coefficient, we integrate equation (38), similarly to equation (30):

$$\Gamma_{\mathrm{V}\infty} \equiv \frac{\alpha_\infty v_0}{\chi_{\mathrm{T}\infty} C_{\mathrm{V}\infty}}, \tag{39}$$

where $\alpha_\infty, \chi_{\mathrm{T}\infty}, C_{\mathrm{V}\infty}$ are the frozen values. Then we get:

$$T - T_0 = T_0\left[e^{\frac{S-S_0}{C_{\mathrm{V}\infty}}}\left(\frac{v}{v_0}\right)^{-\Gamma_{\mathrm{V}\infty}} - 1\right]. \tag{40}$$

Let us obtain the equation of state, which is similar to equation (33):

$$p - p_0 = \frac{1}{\chi_{T\infty}} \left\{ \left(1 - \frac{v}{v_0}\right) + \alpha_\infty T_0 \left[ e^{\frac{S-S_0}{C_{V\infty}}} \left(\frac{v}{v_0}\right)^{-\Gamma_{V\infty}} - 1 \right] \right\}, \qquad (41)$$

where

$$\chi_{T\infty} = v_0 c_{T\infty}^{-2} = \gamma_\infty v_0 c_{S\infty}^{-2}, \qquad (42)$$

$c_{T\infty}$, $c_{S\infty}$ are the isothermal and adiabatic frozen velocities of sound; $\gamma_\infty$ is the frozen polytropic index. If the thermal part prevails above the elastic one

$$\left(1 - \frac{v}{v_0}\right) << \alpha_\infty T_0 \left[ e^{\frac{S-S_0}{C_{V\infty}}} \left(\frac{v}{v_0}\right)^{-\Gamma_{V\infty}} - 1 \right],$$

then equation (41) has the form:

$$p - p_0 = \frac{c_{S\infty}^2 \alpha_\infty T_0}{\gamma_\infty v_0} \left[ \sigma_\infty (S) \left(\frac{v}{v_0}\right)^{-\Gamma_{V\infty}} - 1 \right], \qquad (43)$$

$$\sigma_\infty (S) \equiv \exp\left(\frac{S - S_0}{C_{V\infty}}\right). \qquad (44)$$

For adiabatic processes ($\sigma_\infty (S) = 1,\ (S = S_0)$):

$$p - p_0 = \frac{c_{S\infty}^2 \alpha_\infty T_0}{\gamma_\infty v_0} \left[ \left(\frac{v}{v_0}\right)^{-\Gamma_{V0}} - 1 \right]. \qquad (45)$$

Equations (37) and (45) are the equations of state in the limiting cases of quasistatic (equilibrium) and instant (frozen) processes. It turns out that (45) is similar to Tate's equation (37). From (45) it follows that in this case the sound spreads with frozen velocity $c_{S\infty}$.

# 5 Nonlinear dynamical equation of state with the first and the second time derivatives

Consider the dynamical equation of state (25) with physical nonlinearity (36) [46]:

$$p-p_0=\frac{c_{S0}^2\alpha_0 T_0}{\gamma_0\upsilon_0}\left[\sigma\left(S\right)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]-\tau_{TV}\frac{dp}{dt}-\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt}+\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\frac{dT}{dt}+$$
$$+\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}+\frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2}{\alpha_\infty\chi_{T0}}\tau_{PV}^2\frac{d^2T}{dt^2}+\Gamma\varepsilon_r\left\{\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xx}-\frac{\upsilon_x^2}{\upsilon}\right)+\right.$$
$$\left.+\tau_{TV}\left(p_{xx}-\frac{\upsilon_x}{\upsilon}p_x\right)-\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(T_{xx}-\frac{\upsilon_x}{\upsilon}T_x\right)\right\}. \tag{46}$$

In the variables $(p,\rho,T)$ equation (46) has the form:

$$\frac{1}{\rho^2}\left\{\frac{\Gamma\varepsilon_r}{\tau_{TP}}\left[-\rho_{xx}+\frac{1}{\rho}\left(\rho_x\right)^2\right]+\left[-\frac{d^2\rho}{dt^2}+\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2\right]+\frac{1}{\tau_{TP}}\frac{d\rho}{dt}\right\}+$$
$$+\omega_0^2\sigma\left(s\right)\rho_0^{1-\Gamma_{V0}}\rho^{\Gamma_{V0}}-\omega_0^2\rho_0=b\left(p-p_0\right)+b\tau_{TV}\frac{dp}{dt}-b\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}- \tag{47}$$
$$-b\Gamma\varepsilon_r\tau_{TV}\left(p_{xx}+\frac{\rho_x}{\rho}p_x\right)-\frac{\alpha_\infty^2}{\rho_0\alpha_0}\frac{1}{\tau_{PV}}\frac{dT}{dt}+\frac{\alpha_\infty}{\rho_0}\frac{d^2T}{dt^2}+\frac{\alpha_\infty^2}{\rho_0\alpha_0}\frac{\Gamma\varepsilon_r}{\tau_{PV}}\left(T_{xx}+\frac{\rho_x}{\rho}T_x\right).$$

Here, the following designations are used

$$\omega_0^2\equiv\frac{bc_{S0}^2\alpha_0T_0}{\gamma_0},\qquad b\equiv\frac{\upsilon_0\chi_{T0}}{\tau_{TP}^2},\qquad \sigma\left(S\right)=\exp\left(\frac{S-S_0}{C_{V0}}\right). \tag{48}$$

Including the spatial nonlocality ($\Gamma\varepsilon_r\tau/l^2\equiv l_\varepsilon^2/l^2\Rightarrow 0$) equation (47) transforms into the dynamical equation of state for homogeneous media with physical nonlinearity and temporal nonlocality:

$$\frac{1}{\rho^2}\left\{\left[-\frac{d^2\rho}{dt^2}+\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2\right]+\frac{1}{\tau_{TP}}\frac{d\rho}{dt}\right\}+\omega_0^2\sigma\left(S\right)\rho_0^{1-\Gamma_{V0}}\rho^{\Gamma_{V0}}-\omega_0^2\rho_0=$$
$$=b\left(p-p_0\right)+b\tau_{TV}\frac{dp}{dt}-b\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}-\frac{\alpha_\infty^2}{\rho_0\alpha_0}\frac{1}{\tau_{PV}}\frac{dT}{dt}+\frac{\alpha_\infty}{\rho_0}\frac{d^2T}{dt^2}. \tag{49}$$

A similar equation with variable coefficients taking into account media heterogeneity was obtained in [65]. In the variables $(p, v, T)$, equation (49) has the form:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0 v_0} \left[ \sigma(S) \left( \frac{v}{v_0} \right)^{-\Gamma_{V0}} - 1 \right] - \tau_{TV} \frac{dp}{dt} - \frac{\tau_{TP}}{v_0 \chi_{T0}} \frac{dv}{dt} + \frac{\alpha_0 \tau_{PV}}{\chi_{T0}} \frac{dT}{dt} + \\ + \frac{\chi_{T0}}{\chi_{T\infty}} \tau_{TV}^2 \frac{d^2 p}{dt^2} + \frac{\tau_{TP}^2}{v_0 \chi_{T0}} \frac{d^2 v}{dt^2} - \frac{\alpha_0^2 \tau_{PV}^2}{\chi_{T0} \alpha_\infty} \frac{d^2 T}{dt^2}. \tag{50}$$

If we omit the terms with the second time derivatives in equation (47), then we pass to the dynamical equation of state obtained in [46]:

$$\frac{1}{\rho^2} \left\{ \frac{\Gamma \varepsilon_r}{\tau_{TP}} \left[ -\rho_{xx} + \frac{1}{\rho} (\rho_x)^2 \right] + \frac{1}{\tau_{TP}} \frac{d\rho}{dt} \right\} + \omega_0^2 \sigma(S) \rho_0^{1-\Gamma_{V0}} \rho^{\Gamma_{V0}} - \omega_0^2 \rho_0 = \\ = b(p - p_0) + b \tau_{TV} \frac{dp}{dt} - \frac{\alpha_\infty^2}{\rho_0 \alpha_0} \frac{1}{\tau_{PV}} \frac{dT}{dt} - b \Gamma \varepsilon_r \tau_{TV} \left( p_{xx} + \frac{\rho_x}{\rho} p_x \right) + \\ + \frac{\alpha_\infty^2}{\rho_0 \alpha_0} \frac{\Gamma \varepsilon_r}{\tau_{PV}} \left( T_{xx} + \frac{\rho_x}{\rho} T_x \right). \tag{51}$$

A similar equation is obtained in [43, 44] at the other choice of functions $\sigma(S)$ and coefficient $\omega_0^2$.

In the variables $(p, v, T)$, equation (51) has the following form:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\gamma_0 v_0} \left[ \sigma(S) \left( \frac{v}{v_0} \right)^{-\Gamma_{V0}} - 1 \right] - \tau_{TV} \frac{dp}{dt} - \frac{\tau_{TP}}{v_0 \chi_{T0}} \frac{dv}{dt} + \frac{\alpha_0 \tau_{PV}}{\chi_{T0}} \frac{dT}{dt} + \\ + \Gamma \varepsilon_r \left\{ \tau_{TV} \left( p_{xx} - \frac{v_x}{v} p_x \right) + \frac{\tau_{TP}}{v_0 \chi_{T0}} \left[ v_{xx} - \frac{1}{v} (v_x)^2 \right] - \tau_{PV} \frac{\alpha_0}{\chi_{T0}} \left( T_{xx} - \frac{v_x}{v} T_x \right) \right\}. \tag{52}$$

If we turn off spatial nonlocality ($l_\varepsilon^2 / l^2 \Rightarrow 0$), then equation (52) passes to the dynamical equation of state for nonlinear relaxing media:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{v_0 \gamma_0} \left[ \sigma(S) \left( \frac{v}{v_0} \right)^{-\Gamma_{V0}} - 1 \right] - \tau_{TV} \frac{dp}{dt} - \frac{\tau_{TP}}{v_0 \chi_{T0}} \frac{dv}{dt} + \tau_{PV} \frac{\alpha_0}{\chi_{T0}} \frac{dT}{dt}, \tag{53}$$

which is similar to that obtained in [65].

# 6 Particular models contained in the nonlocal model

It was noted in the previous section that the particular cases of the nonlinear media with internal variables taking into account spatial and temporal nonlocalities (46) are nonlinear media with temporal nonlocality (50), nonlinear relaxing media with spatial nonlocality (52), and nonlinear relaxing media (53).

Let us consider another particular cases of these models for nonlinear relaxing media.

At $\tau_{\mathrm{PV}}\omega \Rightarrow 0$, $\sigma(S) = 1$ ($S = \mathrm{const}$) from equation (53) we obtain **the Lyakhov equation** for nonlinear liquid multicomponent media with volume viscosity (also called bulk viscosity, second viscosity, and dilatational viscosity) [65]:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0} \left[ \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{\mathrm{V}0}} - 1 \right] - \tau_{\mathrm{TV}} \frac{dp}{dt} - \frac{\tau_{\mathrm{TP}}}{\upsilon_0 \chi_{\mathrm{T}0}} \frac{d\upsilon}{dt}, \tag{54}$$

which can be regarded as a nonlinear generalization of **the Oldroyd equation**:

$$p - p_0 = -\tau_{\mathrm{TV}} \frac{dp}{dt} - \frac{\tau_{\mathrm{TP}}}{\upsilon_0 \chi_{\mathrm{T}0}} \frac{d\upsilon}{dt}.$$

Using equation (52), the spatial nonlocal effects can be incorporated into equation (54) :

$$\begin{aligned} p - p_0 = {} & \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0} \left[ \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{\mathrm{V}0}} - 1 \right] - \tau_{\mathrm{TV}} \frac{dp}{dt} - \frac{\tau_{\mathrm{TP}}}{\upsilon_0 \chi_{\mathrm{T}0}} \frac{d\upsilon}{dt} + \\ & + \Gamma \varepsilon_r \left\{ \tau_{\mathrm{TV}} \left( p_{xx} - \frac{\upsilon_x}{\upsilon} p_x \right) + \frac{\tau_{\mathrm{TP}}}{\upsilon_0 \chi_{\mathrm{T}0}} \left[ \upsilon_{xx} - \frac{1}{\upsilon} (\upsilon_x)^2 \right] \right\}. \end{aligned} \tag{55}$$

At $\tau_{\mathrm{PV}}\omega \Rightarrow 0$, $\tau_{\mathrm{TV}}\omega \Rightarrow 0$, $\sigma(S) = 1$ ($S = \mathrm{const}$) from equation (53) we obtain:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0} \left[ \left( \frac{\upsilon}{\upsilon_0} \right)^{-\Gamma_{\mathrm{V}0}} - 1 \right] - \frac{\tau_{\mathrm{TP}}}{\upsilon_0 \chi_{\mathrm{T}0}} \frac{d\upsilon}{dt}. \tag{56}$$

This equation is postulated by Lyakhov at the construction of the equation of state for the relaxing component in multicomponent media [182, 183]. The spatially nonlocal generalization of equation (56) takes on the form:

$$p-p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0}\left[\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}} - 1\right] - \frac{\tau_{TP}}{\upsilon_0 \chi_{T0}}\frac{d\upsilon}{dt} + \Gamma\varepsilon_r\left\{\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left[\upsilon_{xx} - \frac{1}{\upsilon}(\upsilon_x)^2\right]\right\}. \tag{57}$$

At $\tau_{PV}\omega \Rightarrow 0, \tau_{TP}\omega \Rightarrow 0$ from equation (53) we have equation [65]:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}} - 1\right] - \tau_{TV}\frac{dp}{dt}. \tag{58}$$

Equation (58) is the nonlinear generalization of **the Kelvin equation**:

$$p - p_0 = -\tau_{TV}\frac{dp}{dt}.$$

Taking into account the terms corresponding to the spatial nonlocal effects we rewrite equation (58) in the following form:

$$p-p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}} - 1\right] - \tau_{TV}\frac{dp}{dt} + \Gamma\varepsilon_r\left\{\tau_{TV}\left[p_{xx} + \frac{\upsilon}{\upsilon_0}p_x\right]\right\}. \tag{59}$$

If we consider equation (50) instead of equation (53), then, accounting $\tau_{PV}\omega \Rightarrow 0, \tau_{TV}\omega \Rightarrow 0$, we get:

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}} - 1\right] - \frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt} + \frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}. \tag{60}$$

Equation (60) is the nonlinear generalization of the well-known **Kelvin-Voigt equation**:

$$p - p_0 = -\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt} + \frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt}.$$

The spatially nonlocal equation follows from equation (46):

$$p-p_0=\frac{c_{S0}^2\alpha_0T_0}{\upsilon_0\gamma_0}\left[\sigma\left(S\right)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{\mathrm{V0}}}-1\right]-\frac{\tau_{\mathrm{TP}}}{\upsilon_0\chi_{\mathrm{T0}}}\frac{d\upsilon}{dt}+\frac{\tau_{\mathrm{TP}}^2}{\upsilon_0\chi_{\mathrm{T0}}}\frac{d^2\upsilon}{dt^2}+$$
$$+\Gamma\varepsilon_r\left\{\frac{\tau_{\mathrm{TP}}}{\upsilon_0\chi_{\mathrm{T0}}}\left[\upsilon_{xx}-\frac{1}{\upsilon}\left(\upsilon_x\right)^2\right]\right\}. \tag{61}$$

In the limiting cases of quasistatic ($\tau_{\mathrm{PV}}\omega\Rightarrow 0,\tau_{\mathrm{TV}}\omega\Rightarrow 0,\tau_{\mathrm{TP}}\omega\Rightarrow 0$) and instant ($\tau\omega\Rightarrow\infty$ ) processes equation of state (37) and equation of state with frozen relaxing processes (45) follow from equation (53).

# 7 Accounting of temperature terms

In the adiabatic case $S=\mathrm{const}$, we can exclude variable $T$ from the dynamical equation of state (46) with the help of relation (40) and obtain the following equation of state:

$$\frac{1}{\rho^2}\frac{\Gamma\varepsilon_r}{\tau_{\mathrm{TP}}}\left\{\left[-\rho_{xx}\left(1+a\right)+\frac{1}{\rho}\left(\rho_x\right)^2\left(1-a\Gamma_{\mathrm{V0}}\right)\right]+\left[-\frac{d^2\rho}{dt^2}\left(1+a\right)+\right.\right.$$
$$\left.\left.+\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2\left(1-\frac{a\left(\Gamma_{\mathrm{V0}}-1\right)}{2}\right)+\frac{1}{\tau_{\mathrm{TP}}}\frac{d\rho}{dt}\left(1+a\right)\right]\right\}+\omega_0^2\rho_0^{1-\Gamma_{\mathrm{V0}}}\rho^{\Gamma_{\mathrm{V0}}}-$$
$$-\omega_0^2\rho_0=b\left(p-p_0\right)+b\tau_{\mathrm{TV}}\frac{dp}{dt}-\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}b\tau_{\mathrm{TV}}^2\frac{d^2p}{dt^2}-b\Gamma\varepsilon_r\tau_{\mathrm{TV}}\left(p_{xx}+\frac{\rho_x}{\rho}p_x\right), \tag{62}$$

where

$$a\equiv T_0\alpha_\infty\Gamma_{\mathrm{V0}}\left(\frac{\rho}{\rho_0}\right)^{\Gamma_{\mathrm{V0}}+1},\quad \omega_0^2\equiv\frac{bc_{S0}^2\alpha_0T_0}{\gamma_0}. \tag{63}$$

The two-dimensional variant of equation (62) has the form:

$$\frac{1}{\rho^2}\left\{\frac{\Gamma\varepsilon}{\tau_{\mathrm{TP}}}\left[\left(-\Delta\rho\right)\left(1+a\right)+\rho^{-1}\left(\mathrm{grad}\,\rho\right)^2\left(1-a\Gamma_{\mathrm{V0}}\right)\right]+\right.$$
$$\left.+\left[-\frac{d^2\rho}{dt^2}\left(1+a\right)+\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2\left(1-\frac{a\left(\Gamma_{\mathrm{V0}}-1\right)}{2}\right)+\frac{1}{\tau_{\mathrm{TP}}}\frac{d\rho}{dt}\left(1+a\right)\right]\right\}+$$
$$+\omega_0^2\rho_0^{1-\Gamma_{\mathrm{V0}}}\rho^{\Gamma_{\mathrm{V0}}}-\omega_0^2\rho_0=b\left(p-p_0\right)+b\tau_{\mathrm{TV}}\frac{dp}{dt}-\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}b\tau_{\mathrm{TV}}^2\frac{d^2p}{dt^2}-$$
$$-b\Gamma\varepsilon_r\tau_{\mathrm{TV}}\left[\Delta p+\mathrm{grad}\,p\rho^{-1}\left(\mathrm{grad}\,\rho\right)\right].$$

# 8 The nonlinear dynamical equation of state with the third order time derivatives

Suppose that relation (7) between the generalized thermodynamical fluxes and forces takes place. Using the methods stated above for nonlinear media, we obtain the nonlocal dynamical equation of state containing the third order time derivatives. This equation is given in paper [47]:

$$p-p_0=\frac{c_{S0}^2\alpha_0 T_0}{\upsilon_0\gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]-\tau_{TV}\frac{dp}{dt}-\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt}+$$

$$+\frac{\alpha_0\tau_{PV}}{\chi_{T0}}\frac{dT}{dt}+\frac{\chi_{T0}\tau_{TV}^2}{\chi_{T\infty}}\frac{d^2p}{dt^2}+\frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2\tau_{PV}^2}{\chi_{T0}\alpha_\infty}\frac{d^2T}{dt^2}- \tag{64}$$

$$-\left(\frac{\chi_{T0}}{\chi_{T\infty}}\right)^2\tau_{TV}^3\frac{d^3p}{dt^3}-\frac{\tau_{TP}^3}{\upsilon_0\chi_{T0}}\frac{d^3\upsilon}{dt^3}+\frac{\alpha_0^3\tau_{TP}^3}{\chi_{T0}\alpha_\infty^2}\frac{d^3T}{dt^3}+$$

$$+\Gamma\varepsilon_r\left[\tau_{TV}\left(p_{xx}-\frac{\upsilon_x}{\upsilon}p_x\right)+\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xx}-\frac{\upsilon_x^2}{\upsilon}\right)-\frac{\alpha_0\tau_{PV}}{\chi_{T0}}\left(T_{xx}-\frac{\upsilon_x}{\upsilon}T_x\right)\right].$$

In $(p,\rho)$ it has the form:

$$\frac{1}{\rho^2}\frac{\Gamma\varepsilon_r}{\tau_{TP}}\left\{\left[-\rho_{xx}(1-a)+\frac{1}{\rho}(\rho_x)^2(1-a\Gamma_{V0})\right]+\left[-\frac{d^2\rho}{dt^2}(1+a)+\right.\right.$$

$$+\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2\left(1-\frac{a(\Gamma_{V0}-1)}{2}\right)+\left.\frac{1}{\tau_{TP}}\frac{d\rho}{dt}(1+a)\right]+\frac{\tau_{TP}}{\rho^2}\frac{d^3\rho}{dt^3}\left(1+\left(\frac{\alpha_0}{\alpha_\infty}\right)^3a\right)-$$

$$-\frac{3}{\rho}\left(2-\left(\frac{\alpha_0}{\alpha_\infty}\right)^3(\Gamma_{V0}+1)a\right)\frac{d\rho}{dt}\frac{d^2\rho}{dt^2}+ \tag{65}$$

$$+\frac{1}{\rho^2}\left(6-\left(\frac{\alpha_0}{\alpha_\infty}\right)^3(\Gamma_{V0}+1)(\Gamma_{V0}+2)a\right)\left(\frac{d\rho}{dt}\right)^3+\omega_0^2\rho_0^{1-\Gamma_{V0}}\rho_0^{\Gamma_{V0}}-\omega_0^2\rho_0=$$

$$=b(p-p_0)+b_{TV}\frac{dp}{dt}-\frac{\chi_{T0}}{\chi_{T\infty}}b\tau_{TV}^2\frac{d^2p}{dt^2}+\left(\frac{\chi_{T0}}{\chi_{T\infty}}\right)^2b\tau_{TV}^3\frac{d^3p}{dt^3}-b\Gamma\varepsilon_r\tau_{TV}\left(p_{xx}+\frac{\rho_x}{\rho}p_x\right).$$

# 9 The nonlinear dynamical equation of state with the third time and spatial derivatives taking into account the temperature terms and the unsteady thermodynamical forces

Now, let us consider the case when thermodynamical fluxes and forces are both unsteady, and phenomenological relation (6) takes place. Then we can obtain the following nonlinear dynamical equation of state [48]:

$$p-p_0=\frac{c_{S0}^2\alpha_0 T_0}{\upsilon_0\gamma_0}\left[\sigma\left(S\right)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{\mathrm{V0}}}-1\right]-\left(1-\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\right)\tau_{\mathrm{TV}}\frac{dp}{dt}+$$

$$+\left(\frac{\alpha_0}{\chi_{\mathrm{T0}}}-\frac{\alpha_0^2}{\chi_{\mathrm{T0}}\alpha_\infty}\right)\tau_{\mathrm{PV}}\frac{dT}{dt}+\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\tau_{\mathrm{TV}}^2\frac{d^2p}{dt^2}+\frac{\tau_{\mathrm{TP}}^2}{\upsilon_0\chi_{\mathrm{T0}}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2}{\chi_{\mathrm{T0}}\alpha_\infty}\tau_{\mathrm{PV}}^2\frac{d^2T}{dt^2}-$$

$$-\left(\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\right)^2\tau_{\mathrm{TV}}^3\frac{d^3p}{dt^3}-\frac{\tau_{\mathrm{TP}}^3}{\upsilon_0\chi_{\mathrm{T0}}}\frac{d^3\upsilon}{dt^3}+\frac{\alpha_0^3}{\chi_{\mathrm{T0}}\alpha_\infty^2}\tau_{\mathrm{PV}}^3\frac{d^3T}{dt^3}+$$

$$+\Gamma\varepsilon_r\left[\tau_{\mathrm{TV}}\left(p_{xx}-\frac{\upsilon_x}{\upsilon}p_x\right)+\frac{\tau_{\mathrm{TP}}}{\upsilon_0\chi_{\mathrm{T0}}}\left(\upsilon_{xx}-\frac{\left(\upsilon_x\right)^2}{\upsilon}\right)-\right.$$

$$\left.-\frac{\alpha_0}{\chi_{\mathrm{T0}}}\tau_{\mathrm{PV}}\left(T_{xx}-\frac{\upsilon_x}{\upsilon}T_x\right)\right]-\Gamma\varepsilon_r\tau_{\mathrm{TP}}\left\{\tau_{\mathrm{TV}}\left[\left(p_{xxt}-\frac{\upsilon_x p_{xt}}{\upsilon}-\frac{p_x\upsilon_{xt}}{\upsilon}+\frac{\upsilon_x\upsilon_t p_x}{\upsilon^2}\right)+\right.\right.$$

$$\left.\left.u\left(p_{xxx}-\frac{\upsilon_x p_{xx}}{\upsilon}-\frac{p_x\upsilon_{xx}}{\upsilon}+\frac{\left(\upsilon_x\right)^2 p_x}{\upsilon^2}\right)\right]+\right.$$

$$+\frac{\tau_{\mathrm{TP}}}{\upsilon_0\chi_{\mathrm{T0}}}\left[\left(\upsilon_{xxt}-\frac{2\upsilon_x\upsilon_{xt}}{\upsilon}+\frac{\left(\upsilon_x\right)^2\upsilon_t}{\upsilon^2}\right)+u\left(\upsilon_{xxx}-\frac{2\upsilon_x\upsilon_{xx}}{\upsilon}+\frac{\left(\upsilon_x\right)^3}{\upsilon^2}\right)\right]-$$

$$\left.-\frac{\alpha_0\tau_{\mathrm{PV}}}{\chi_{\mathrm{T0}}}\left[\left(T_{xxt}-\frac{\upsilon_x T_{xt}}{\upsilon}-\frac{T_x\upsilon_{xt}}{\upsilon}+\frac{\upsilon_x\upsilon_t T_x}{\upsilon^2}\right)+u\left(T_{xxx}-\frac{\upsilon_x T_{xx}}{\upsilon}-\frac{T_x\upsilon_{xx}}{\upsilon}+\frac{\left(\upsilon_x\right)^2 T_x}{\upsilon^2}\right)\right]\right\},$$

where $u$ is the mass velocity.

# 10 The dynamical equation of state taking into account both elastic and thermal components of internal energy (the second order time and spatial derivatives)

Consider the dynamical equation of state (25) with the physical nonlinearity defined by (32)

$$p-p_0=\frac{c_{S0}^2}{\gamma_0\upsilon_0}\left(1-\frac{\upsilon}{\upsilon_0}\right)+\frac{c_{S0}^2\alpha_0T_0}{\upsilon_0\gamma_0}\left[\sigma\left(S\right)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]-\tau_{TV}\frac{dp}{dt}-\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt}+$$

$$+\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\frac{dT}{dt}+\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}+\frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2}{\chi_{T0}\alpha_\infty}\tau_{PV}^2\frac{d^2T}{dt^2}+ \tag{66}$$

$$+\Gamma\varepsilon_r\left[\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xx}+\rho^{-1}\rho_x\upsilon_x\right)+\tau_{TV}\left(p_{xx}+\rho^{-1}\rho_xp_x\right)-\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(T_{xx}+\rho^{-1}\rho_xT_x\right)\right].$$

We can say that this relation is the dynamical equation of state for nonlocal media taking into account both the elastic and thermal components of pressure. If we take into account the temperature terms in both dynamical and spatially nonlocal parts of equation (66) (similar to paper [48]) then this equation in variables $(p,\rho)$ takes the form:

$$\frac{1}{\rho^2}\left\{\frac{\Gamma\varepsilon_r}{\tau_{TP}}\left[-\rho_{xx}\left(1+a\right)+\frac{1}{\rho}\left(\rho_x\right)^2\left(1-a\Gamma_{V0}\right)\right]+\left[-\frac{d^2\rho}{dt^2}\left(1+a\right)+\right.\right.$$

$$\left.\left.+\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2\left(1-\frac{a\left(\Gamma_{V0}-1\right)}{2}\right)+\frac{1}{\tau_{TP}}\frac{d\rho}{dt}\left(1+a\right)\right]\right\}+\left(\tilde{\omega}_0^2-\omega_0^2\right)\rho_0-\tilde{\omega}_0^2\rho_0^2\rho^{-1}+$$

$$+\omega_0^2\rho_0^{(1-\Gamma_{V0})}\rho^{\Gamma_{V0}}=b\left(p-p_0\right)+b\tau_{TV}\frac{dp}{dt}-\frac{\chi_{T0}}{\chi_{T\infty}}b\tau_{TV}^2\frac{d^2p}{dt^2}- \tag{67}$$

$$-b\Gamma\varepsilon_r\tau_{TV}\left(p_{xx}+\frac{\rho_x}{\rho}p_x\right),$$

where $\tilde{\omega}_0^2\equiv\dfrac{bc_{S0}^2}{\gamma_0}$.

The particular case of equation (66) is dynamical equation of state (46), obtained in [45, 46]:

$$p-p_0=\frac{c_{S0}^2\alpha_0T_0}{\upsilon_0\gamma_0}\left[\sigma\left(S\right)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]-\tau_{TV}\frac{dp}{dt}-\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt}+\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\frac{dT}{dt}+$$

$$+\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}+\frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2}{\alpha_\infty\chi_{T0}}\tau_{PV}^2\frac{d^2T}{dt^2}+ \tag{68}$$

$$+\Gamma\varepsilon_r\left[\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xx}+\rho^{-1}\rho_x\upsilon_x\right)+\tau_{TV}\left(p_{xx}+\rho_x\rho^{-1}p_x\right)-\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(T_{xx}+\rho^{-1}\rho_xT_x\right)\right].$$

If the elastic part prevails above the thermal one:

$$\alpha_0T_0\left[\sigma\left(S\right)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]<<\left(1-\frac{\upsilon}{\upsilon_0}\right),$$

then from equation (66) it follows the physically linear dynamical equation of state for nonlocal media:

$$p-p_0=\frac{c_{S0}^2}{\gamma_0\upsilon_0}\left(1-\frac{\upsilon}{\upsilon_0}\right)-\tau_{TV}\frac{dp}{dt}-\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt}+\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\frac{dT}{dt}+$$

$$+\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}+\frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2}{\alpha_\infty\chi_{T0}}\tau_{PV}^2\frac{d^2T}{dt^2}+ \tag{69}$$

$$+\Gamma\varepsilon_r\left[\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xx}+\rho^{-1}\rho_x\upsilon_x\right)+\tau_{TV}\left(p_{xx}+\rho^{-1}\rho_xp_x\right)-\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(T_{xx}+\rho^{-1}\rho_xT_x\right)\right].$$

In the limiting case, when we deal with the adiabatic ($\sigma\left(S\right)=1$) quasistatic processes at $\Gamma_{V0}=1$ and omit the spatial nonlocality in nonlinear equation (68), we get:

$$p-p_0=c^2\left(\rho-\rho_0\right)\ \text{ or }\ p=c^2\rho \tag{70}$$

where $c^2=\dfrac{c_{S0}^2\alpha_0T_0}{\gamma_0}$. In the same approximation equation (69) has the form:

$$p=-A^2\upsilon+B, \tag{71}$$

where $A^2 \equiv \frac{c_{S0}^2}{\gamma_0 \upsilon_0^2}$, $B \equiv p_0 + \frac{c_{S0}^2}{\gamma_0 \upsilon_0^2}$. Equations (71) and (71) are considered in papers [182, 183] by Lyakhov.

The generalized dynamical equation of state for media with temporal nonlocality can be obtained from the equations presented above omitting the spatial nonlocality ($\varepsilon_r = 0$). For media with temporal nonlocality in the case of steady thermodynamical forces ($\varepsilon_r = 0$, $\mathrm{K} = 0$), when the generalized phenomenological relation (13) takes place, the corresponding dynamical equation of state of the N-th order has the form

$$p - p_0 = \frac{c_{S0}^2}{\gamma_0 \upsilon_0}\left(1 - \frac{\upsilon}{\upsilon_0}\right) + \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{\mathrm{V0}}} - 1\right] - \\ - \sum_{n=0}^{\mathrm{N-1}} (-1)^n \frac{d^n}{dt^n}\left[\left(\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\right)^n \tau_{\mathrm{TV}}^{(\mathrm{n}+1)} \frac{dp}{dt} + \frac{\tau_{\mathrm{TP}}^{(\mathrm{n}+1)}}{\upsilon_0 \chi_{\mathrm{T0}}} \frac{d\upsilon}{dt} - \frac{\alpha_0^{(n+1)}}{\chi_{\mathrm{T0}} \alpha_\infty^n} \tau_{\mathrm{PV}}^{(\mathrm{n}+1)} \frac{dT}{dt}\right]. \tag{72}$$

# 11 Generalization of the dynamical equations of state for media with temporal and spatial nonlocalities

In paper [49], using constitutive relation (4), it has been obtained the dynamical equation of state for media with internal oscillators

$$p - p_0 = \frac{c_{S0}^2 \alpha_0 T_0}{\upsilon_0 \gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{\mathrm{V0}}} - 1\right] - \\ - \sum_{n=0}^{\mathrm{N}} (-1)^n \frac{d^n}{dt^n}\left[\left(\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\right)^n \tau_{\mathrm{TV}}^{(n+1)} \frac{dp}{dt} + \frac{\tau_{\mathrm{TP}}^{(n+1)}}{\upsilon_0 \chi_{\mathrm{T0}}} \frac{d\upsilon}{dt} - \frac{\alpha_0^{(n+1)}}{\chi_{\mathrm{T0}} \alpha_\infty^n} \tau_{\mathrm{PV}}^{(n+1)} \frac{dT}{dt}\right] - \\ - \sum_{k=1}^{\mathrm{K}} (-1)^k \frac{d^k}{dt^k}\left[\left(\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\right)^k \tau_{\mathrm{TV}}^k p + \frac{\tau_{\mathrm{TP}}^k}{\upsilon_0 \chi_{\mathrm{T0}}} \upsilon - \frac{\alpha_0^{(k+1)}}{\chi_{\mathrm{T0}} \alpha_\infty^{(k-1)}} \tau_{\mathrm{PV}}^k T\right] + \tag{73}$$
$$+ \Gamma \varepsilon_r \sum_{k=0}^{\mathrm{K}} (-1)^k \frac{d^k}{dt^k}\left[\left(\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\right)^k \tau_{\mathrm{TV}}^{(k+1)}\left(p_{xx} - \frac{\upsilon_x}{\upsilon} p_x\right) + \frac{\tau_{\mathrm{TP}}^{(k+1)}}{\upsilon_0 \chi_{\mathrm{T0}}}\left(\upsilon_{xx} - \frac{\upsilon_x^2}{\upsilon}\right) -$$

$$- \frac{\alpha_0^{(k+1)}}{\chi_{\mathrm{T}0}\alpha_\infty^k}\tau_{\mathrm{PV}}^{(k+1)}\left(T_{xx} - \frac{\upsilon_x}{\upsilon}T_x\right)\Bigg].$$

Taking into account both the elastic and the thermal pressure components in constitutive relation (4), the dynamical equation of state can be written in the general form

$$p - p_0 = \frac{c_{S0}^2}{\gamma_0\upsilon_0}\left(1 - \frac{\upsilon}{\upsilon_0}\right) + \frac{c_{S0}^2\alpha_0 T_0}{\upsilon_0\gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{\mathrm{V}0}} - 1\right] -$$

$$- \sum_{n=0}^{\mathrm{N}}(-1)^n\frac{d^n}{dt^n}\left[\left(\frac{\chi_{\mathrm{T}0}}{\chi_{\mathrm{T}\infty}}\right)^n\tau_{\mathrm{TV}}^{(n+1)}\frac{dp}{dt} + \frac{\tau_{\mathrm{TP}}^{(n+1)}}{\upsilon_0\chi_{\mathrm{T}0}}\frac{d\upsilon}{dt} - \frac{\alpha_0^{(n+1)}}{\chi_{\mathrm{T}0}\alpha_\infty^n}\tau_{\mathrm{PV}}^{(n+1)}\frac{dT}{dt}\right] -$$

$$- \sum_{k=1}^{\mathrm{K}}(-1)^k\frac{d^k}{dt^k}\left[\left(\frac{\chi_{\mathrm{T}0}}{\chi_{\mathrm{T}\infty}}\right)^k\tau_{\mathrm{TV}}^k p + \frac{\tau_{\mathrm{TP}}^k}{\upsilon_0\chi_{\mathrm{T}0}}\upsilon - \frac{\alpha_0^{(k+1)}}{\chi_{\mathrm{T}0}\alpha_\infty^k}\tau_{\mathrm{PV}}^k T\right] + \qquad (74)$$

$$+\Gamma\varepsilon_r\sum_{k=0}^{\mathrm{K}}(-1)^k\frac{d^k}{dt^k}\left[\left(\frac{\chi_{\mathrm{T}0}}{\chi_{\mathrm{T}\infty}}\right)^k\tau_{\mathrm{TV}}^{(k+1)}\left(p_{xx} - \frac{\upsilon_x}{\upsilon}p_x\right) + \frac{\tau_{\mathrm{TP}}^{(k+1)}}{\upsilon_0\chi_{\mathrm{T}0}}\left(\upsilon_{xx} - \frac{\upsilon_x^2}{\upsilon}\right) -\right.$$

$$\left. - \frac{\alpha_0^{(k+1)}}{\chi_{\mathrm{T}0}\alpha_\infty^k}\tau_{\mathrm{PV}}^{(k+1)}\left(T_{xx} - \frac{\upsilon_x}{\upsilon}T_x\right)\right].$$

If constitutive relation (3) holds and the thermal and elastic pressure components are taken into account, then the generalized dynamical equation of state for media with internal oscillators takes the form

$$p - p_0 = \frac{c_{S0}^2}{\gamma_0\upsilon_0}\left(1 - \frac{\upsilon}{\upsilon_0}\right) + \frac{c_{S0}^2\alpha_0 T_0}{\upsilon_0\gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{\mathrm{V}0}} - 1\right] -$$

$$- \sum_{n=0}^{\mathrm{N}}(-1)^n\frac{d^n}{dt^n}\left[\left(\frac{\chi_{\mathrm{T}0}}{\chi_{\mathrm{T}\infty}}\right)^n\tau_{\mathrm{TV}}^{(n+1)}\frac{dp}{dt} + \frac{\tau_{\mathrm{TP}}^{(n+1)}}{\upsilon_0\chi_{\mathrm{T}0}}\frac{d\upsilon}{dt} - \frac{\alpha_0^{(n+1)}}{\chi_{\mathrm{T}0}\alpha_\infty^n}\tau_{\mathrm{PV}}^{(n+1)}\frac{dT}{dt}\right] -$$

$$- \sum_{k=1}^{\mathrm{K}}(-1)^k\frac{d^k}{dt^k}\left[\left(\frac{\chi_{\mathrm{T}0}}{\chi_{\mathrm{T}\infty}}\right)^k\tau_{\mathrm{TV}}^k p + \frac{\tau_{\mathrm{TP}}^k}{\upsilon_0\chi_{\mathrm{T}0}}\upsilon - \frac{\alpha_0^{(k+1)}}{\chi_{\mathrm{T}0}\alpha_\infty^k}\tau_{\mathrm{PV}}^k T\right] + \qquad (75)$$

$$+\Gamma\varepsilon_r \sum_{k=0}^{\mathrm{K}} (-1)^k \frac{d^k}{dt^k} \sum_{m=0}^{\mathrm{M}} (-1)^{(m+1)} \left(\frac{\upsilon}{\upsilon_x}\right)^{(m-1)} \frac{d^{(m+1)}}{dx^{(m+1)}} \left[ \left(\frac{\chi_{\mathrm{T0}}}{\chi_{\mathrm{T}\infty}}\right)^k \tau_{\mathrm{TV}}^{(k+1)} p + \right.$$

$$\left. + \frac{\tau_{\mathrm{TP}}^{(k+1)}}{\upsilon_0 \chi_{\mathrm{T0}}} \upsilon - \frac{\alpha_0^{(k+1)}}{\chi_{\mathrm{T0}} \alpha_\infty^k} \tau_{\mathrm{PV}}^{(k+1)} T \right].$$

# 12 The complete quasistatic equation of state

Analysing the conditions of thermodynamical stability we obtain the following quasistatic equation of state

$$p = \frac{c_{S0}^2}{\gamma_0 \upsilon_0} \left[ \frac{(1-\gamma_0)}{3\Gamma_{\mathrm{V0}}} \left(\frac{\upsilon}{\upsilon_0}\right)^{-1} + \frac{(\gamma_0 - 1)}{\Gamma_{\mathrm{V0}}} \sigma(S) \left(\frac{\upsilon}{\upsilon_0}\right)^{-(\Gamma_{\mathrm{V0}}+1)} \right]. \qquad (76)$$

If we assume $p = p_0$, $\upsilon = \upsilon_0$, $S = S_0$ then from (76) it follows that the equilibrium value of $p_0$ is

$$p_0 = \frac{c_{S0}^2}{\gamma_0 \upsilon_0} \left[ \frac{(1-\gamma_0)}{3\Gamma_{\mathrm{V0}}} + \frac{(\gamma_0 - 1)}{\Gamma_{\mathrm{V0}}} \right] \qquad (77)$$

and the complete quasistatic equation of state is

$$p - p_0 = \frac{c_{S0}^2}{\gamma_0 \upsilon_0} \left\{ \frac{(1-\gamma_0)}{3\Gamma_{\mathrm{V0}}} \left[ \left(\frac{\upsilon}{\upsilon_0}\right)^{-1} - 1 \right] + \frac{(\gamma_0 - 1)}{\Gamma_{\mathrm{V0}}} \left[ \sigma(S) \left(\frac{\upsilon}{\upsilon_0}\right)^{-(\Gamma_{\mathrm{V0}}+1)} - 1 \right] \right\}. \qquad (78)$$

Let us carry out the linearization of equation (78) in the vicinity of the equilibrium state $(\rho_0, p_0)$. Let

$$\upsilon = \upsilon_0 + \varepsilon \upsilon_1 + \varepsilon^2 \upsilon_2 + ..., \qquad p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + ..., \qquad (79)$$

$$\upsilon^n \approx \upsilon_0^n + n\upsilon_0^{(n-1)} \varepsilon \upsilon_1 + n\upsilon_0^{(n-1)} \varepsilon^2 \upsilon_2 + \frac{(n-1)\,n}{2!} \upsilon_0^{(n-2)} \upsilon_1^2 \varepsilon^2. \qquad (80)$$

Using the expansion in series

$$\left(\frac{\upsilon}{\upsilon_0}\right)^{-1} = 1 - \varepsilon \frac{\upsilon_1}{\upsilon_0} - \varepsilon^2 \frac{\upsilon_2}{\upsilon_0} + \varepsilon^2 \left(\frac{\upsilon_1}{\upsilon_0}\right)^2 + ...$$

in the first order approximation we get

$$\frac{(1-\gamma_0)}{3\Gamma_{V0}}\left[\left(\frac{v}{v_0}\right)^{-1}-1\right]=\frac{(1-\gamma_0)}{3\Gamma_{V0}}\left(1-\frac{v}{v_0}\right). \tag{81}$$

In the same way we have

$$\left(\frac{v}{v_0}\right)^{-(\Gamma_{V0}+1)}=\left(\frac{v_0+\varepsilon v_1}{v_0}\right)^{-(\Gamma_{V0}+1)}=1-(\Gamma_{V0}+1)\frac{\varepsilon v_1}{v_0}=$$

$$=1-(\Gamma_{V0}+1)\left(\frac{v}{v_0}-1\right)=1+\Gamma_{V0}\left(1-\frac{v}{v_0}\right)+\left(1-\frac{v}{v_0}\right); \tag{82}$$

$$\left(\frac{v}{v_0}\right)^{-\Gamma_{V0}}=\left(1+\frac{\varepsilon v_1}{v_0}\right)^{-\Gamma_{V0}}=1-\Gamma_{V0}\left(\frac{v}{v_0}-1\right)=1+\Gamma_{V0}\left(1-\frac{v}{v_0}\right)\Rightarrow$$

$$\frac{(\gamma_0-1)}{\Gamma_{V0}}\left[\sigma(S)\left(\frac{v}{v_0}\right)^{-(\Gamma_{V0}+1)}-1\right]=$$

$$=\frac{(\gamma_0-1)}{\Gamma_{V0}}\left[\sigma(S)\left(\frac{v}{v_0}\right)^{-\Gamma_{V0}}-1\right]+\frac{(\gamma_0-1)}{\Gamma_{V0}}\sigma(S)\left(1-\frac{v}{v_0}\right). \tag{83}$$

Thus, in the first order approximation, equation of state (78) has the form

$$p-p_0=\frac{c_{S0}^2}{\gamma_0 v_0}\left\{\left[\frac{(1-\gamma_0)}{\Gamma_{V0}}\left(\frac{1}{3}-\sigma(S)\right)\right]\left(1-\frac{v}{v_0}\right)+\alpha_0 T_0\left[\sigma(S)\left(\frac{v}{v_0}\right)^{-\Gamma_{V0}}-1\right]\right\}, \tag{84}$$

where the formula $\gamma_0-1=\alpha_0 T_0\Gamma_{V0}$ is used.

If the following condition holds:

$$\left[\frac{(1-\gamma_0)}{\Gamma_{V0}}\left(\frac{1}{3}-\sigma(S)\right)\right]\to 1, \tag{85}$$

then equation of state (84) (or (78) in the first order approximation) coincides with the quasistatic part of the dynamical equation of state with spatial nonlocality

$$p-p_0=\frac{c_{S0}^2}{\gamma_0 v_0}\left\{\left(1-\frac{v}{v_0}\right)+\alpha_0 T_0\left[\sigma(S)\left(\frac{v}{v_0}\right)^{-\Gamma_{V0}}-1\right]\right\}. \tag{86}$$

# 13 The construction of the complete dynamical equation of state for media with the temporal nonlocality and the nonlinear dynamical part

In the investigations, described above, the equation of state with temporal nonlocality was obtained

$$p-p_0=\left[\frac{1}{\chi_{\mathrm{T}0}}\left(1-\frac{v}{v_0}\right)+\frac{\alpha_0}{\chi_{\mathrm{T}0}}\left(T-T_0\right)\right]-\tau_{\mathrm{TV}}\frac{dp}{dt}-\frac{\tau_{\mathrm{TP}}}{v_0\chi_{\mathrm{T}0}}\frac{dv}{dt}+\frac{\alpha_0}{\chi_{\mathrm{T}0}}\tau_{\mathrm{PV}}\frac{dT}{dt}. \tag{87}$$

It was shown in [45, 46] that after the change of variables from $(p, v, T)$ to $(p, v, S)$, equation (87) takes the form

$$p-p_0=\frac{c_{S0}^2}{\gamma_0 v_0}\left(1-\frac{v}{v_0}\right)+\frac{c_{S0}^2\alpha_0 T_0}{v_0\gamma_0}\left[\sigma\left(S\right)\left(\frac{v}{v_0}\right)^{-\Gamma_{\mathrm{V}0}}-1\right]-\tau_{\mathrm{TV}}\frac{dp}{dt}-$$
$$-\frac{\tau_{\mathrm{TP}}}{v_0\chi_{\mathrm{T}0}}\frac{dv}{dt}+\frac{\alpha_0}{\chi_{\mathrm{T}0}}\tau_{\mathrm{PV}}\frac{dT}{dt}. \tag{88}$$

Let us show that the dynamical parts of equations (87) and (88) can be constructed by differentiating the quasistatic parts of these equations. Indeed, the differentiation of the quasistatic part of equation (87) with respect to time yields

$$-\frac{dp}{dt}-\frac{1}{v_0\chi_{\mathrm{T}0}}\frac{dv}{dt}+\frac{\alpha_0}{\chi_{\mathrm{T}0}}\frac{dT}{dt}. \tag{89}$$

Multiplying (89) by corresponding times of relaxation we obtain the dynamical part of equation (87)

$$-\tau_{\mathrm{TV}}\frac{dp}{dt}-\frac{\tau_{\mathrm{TP}}}{v_0\chi_{\mathrm{T}0}}\frac{dv}{dt}+\frac{\alpha_0}{\chi_{\mathrm{T}0}}\tau_{\mathrm{PV}}\frac{dT}{dt}. \tag{90}$$

In the same way, differentiating the quasistatic part of equation (88) and using the formula

$$\frac{dT}{dt}=-T_0\Gamma_{\mathrm{V}0}\sigma\left(S\right)\frac{v_0^{\Gamma_{\mathrm{V}0}}}{v^{(\Gamma_{\mathrm{V}0}+1)}}\frac{dv}{dt} \tag{91}$$

it easy to get the dynamical part of equation (88).

Doing in the same spirit let us construct the complete dynamical equation of state with the help of quasistatic equation (78).

First of all we must differentiate equation (78). After multiplying by the times of relaxation, we include the result in equation (78). So that we get

$$p-p_0=\frac{c_{S0}^2}{\gamma_0\upsilon_0}\left\{\frac{(1-\gamma_0)}{3\Gamma_{V0}}\left[\left(\frac{\upsilon}{\upsilon_0}\right)^{-1}-1\right]+\frac{(\gamma_0-1)}{\Gamma_{V0}}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-(\Gamma_{V0}+1)}-1\right]\right\}-$$
$$-\tau_{TV}\frac{dp}{dt}-\frac{c_{S0}^2}{\gamma_0\upsilon_0}\left[\frac{(1-\gamma_0)}{3\Gamma_{V0}}\frac{\upsilon_0}{\upsilon^2}+\frac{(\gamma_0-1)}{\Gamma_{V0}}\sigma(S)\frac{(\Gamma_{V0}+1)\,\upsilon_0^{(\Gamma_{V0}+1)}}{\upsilon^{(\Gamma_{V0}+2)}}\right]\tau_{TP}\frac{d\upsilon}{dt}. \tag{92}$$

Equation (92) is the complete dynamical equation of state for media with temporal nonlocality and the nonlinear dynamical part (the thermal terms are included).

Let us state the conditions under which equation (92) coincides with equation (88). Transforming the dynamical part of equation (92) with the help of formulae (91) and $\gamma_0-1=\alpha_0T_0\Gamma_{V0}$ we get

$$p-p_0=\frac{c_{S0}^2}{\gamma_0\upsilon_0}\left\{\frac{(1-\gamma_0)}{3\Gamma_{V0}}\left[\left(\frac{\upsilon}{\upsilon_0}\right)^{-1}-1\right]+\frac{(\gamma_0-1)}{\Gamma_{V0}}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-(\Gamma_{V0}+1)}-1\right]\right\}-$$
$$-\tau_{TV}\frac{dp}{dt}-\frac{(1-\gamma_0)}{\upsilon_0\chi_{T0}\Gamma_{V0}}\left[\frac{1}{3}-\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}\right]\left(\frac{\upsilon}{\upsilon_0}\right)^{-2}\tau_{TP}\frac{d\upsilon}{dt}+ \tag{93}$$
$$+\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(\frac{\upsilon}{\upsilon_0}\right)^{-1}\frac{dT}{dt}.$$

Taking into account that

$$\left(\frac{\upsilon}{\upsilon_0}\right)^{-1}=2-\frac{\upsilon}{\upsilon_0};\quad\left(\frac{\upsilon}{\upsilon_0}\right)^{-2}=\left(1+\frac{\varepsilon\upsilon_1}{\upsilon_0}\right)^{-2}=1-2\frac{\upsilon}{\upsilon_0};$$
$$\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}=1+\Gamma_{V0}\left(1-\frac{\upsilon}{\upsilon_0}\right),$$

and the quasistatic part of equation (93) has the form (84), we write equation (93) in the first approximation

$$p-p_0=\frac{c_{S0}^2}{\gamma_0\upsilon_0}\left\{\left[\frac{(1-\gamma_0)}{\Gamma_{V0}}\left(\frac{1}{3}-\sigma(S)\right)\right]\left(1-\frac{\upsilon}{\upsilon_0}\right)+\alpha_0T_0\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]\right\}-$$
$$-\tau_{TV}\frac{dp}{dt}-\frac{(1-\gamma_0)}{\upsilon_0\chi_{T0}\Gamma_{V0}}\left[\frac{1}{3}-\sigma(S)-\sigma(S)\Gamma_{V0}\left(1-\frac{\upsilon}{\upsilon_0}\right)\right]\left(3-2\frac{\upsilon}{\upsilon_0}\right)\tau_{TP}\frac{d\upsilon}{dt}+$$
$$+\frac{\alpha_0}{\chi_{T0}}\left(3-2\frac{\upsilon}{\upsilon_0}\right)\tau_{PV}\frac{dT}{dt}. \tag{94}$$

Equation (94) is the complete dynamical equation of state with the nonlinear dynamical part in the first approximation. If the following conditions hold:

$$\frac{\upsilon}{\upsilon_0}\to 1,\frac{(1-\gamma_0)}{\Gamma_{V0}}\left(\frac{1}{3}-\sigma(S)\right)\to 1,$$

then equation (94) coincides with equation of state (88).

# 14 Dynamical equations of state taking into account elastic and thermal pressure components with the third order time and spatial derivatives

Note that, increasing of orders of derivatives in dynamical equations of state corresponds to the enlargement of measure of the deviation from local equilibrium. In this section we are going to consider the dynamical equations of state taking into account elastic and thermal pressure components with the third order time and spatial derivatives.

Using the methods described above and starting from phenomenological constitutive relation (6)

$$\frac{d\xi}{dt}=\Gamma\left[A+\varepsilon_r\left(\xi_{xx}-\frac{\upsilon_x}{\upsilon}\xi_x\right)\right]-\Gamma\tau_{TP}\frac{d}{dt}\left[A+\varepsilon_r\left(\xi_{xx}-\frac{\upsilon_x}{\upsilon}\xi_x\right)\right]+\tau_{TP}\frac{d^2\xi}{dt^2}-$$
$$-\tau_{TP}^2\frac{d^3\xi}{dt^3},$$

we obtain the dynamical equation of state

$$p-p_0=\frac{c_{S0}^2}{\upsilon_0\gamma_0}\left(1-\frac{\upsilon}{\upsilon_0}\right)+\frac{c_{S0}^2\alpha_0 T_0}{\upsilon_0\gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]-\left(1-\frac{\chi_{T0}}{\chi_{T\infty}}\right)\tau_{TV}\frac{dp}{dt}+$$

$$+\left(\frac{\alpha_0}{\chi_{T0}}-\frac{\alpha_0^2}{\chi_{T0}\alpha_\infty}\right)\tau_{PV}\frac{dT}{dt}+\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}+\frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2}{\chi_{T0}\alpha_\infty}\tau_{PV}^2\frac{d^2T}{dt^2}-$$

$$-\left(\frac{\chi_{T0}}{\chi_{T\infty}}\right)^2\tau_{TV}^3\frac{d^3p}{dt^3}-\frac{\tau_{TP}^3}{\upsilon_0\chi_{T0}}\frac{d^3\upsilon}{dt^3}+\frac{\alpha_0^3}{\chi_{T0}\alpha_\infty^2}\tau_{PV}^3\frac{d^3T}{dt^3}+ \qquad (95)$$

$$+\Gamma\varepsilon_r\left[\tau_{TV}\left(p_{xx}-\frac{\upsilon_x}{\upsilon}p_x\right)+\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xx}-\frac{(\upsilon_x)^2}{\upsilon}\right)-\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(T_{xx}-\frac{\upsilon_x}{\upsilon}T_x\right)\right]-$$

$$-\Gamma\varepsilon_r\tau_{TP}\left\{\tau_{TV}\left(p_{xxt}-\frac{\upsilon_x p_{xt}}{\upsilon}-\frac{\upsilon_{xt}p_x}{\upsilon}+\frac{\upsilon_x\upsilon_t p_x}{\upsilon^2}\right)+\right.$$

$$\left.+\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left(\upsilon_{xxt}-\frac{2\upsilon_x\upsilon_{xt}}{\upsilon}+\frac{(\upsilon_x)^2\upsilon_t}{\upsilon^2}\right)-\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\left(T_{xxt}-\frac{\upsilon_x T_{xt}}{\upsilon}-\frac{\upsilon_{xt}T_x}{\upsilon}+\frac{\upsilon_x\upsilon_t T_x}{\upsilon^2}\right)\right\}.$$

If we use the phenomenological constitutive relation

$$\frac{d\xi}{dt}=\Gamma\left[A+\varepsilon_r(-\frac{\upsilon_x}{\upsilon}\xi_x+\xi_{xx}-\frac{\upsilon}{\upsilon_x}\xi_{xxx})\right]+\tau_{TP}\frac{d^2\xi}{dt^2}-\tau_{TP}^2\frac{d^3\xi}{dt^3}.$$

then we get the following dynamical equation of state

$$p-p_0=\frac{c_{S0}^2}{\upsilon_0\gamma_0}\left(1-\frac{\upsilon}{\upsilon_0}\right)+\frac{c_{S0}^2\alpha_0 T_0}{\upsilon_0\gamma_0}\left[\sigma(S)\left(\frac{\upsilon}{\upsilon_0}\right)^{-\Gamma_{V0}}-1\right]-\tau_{TV}\frac{dp}{dt}-\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\frac{d\upsilon}{dt}+$$

$$+\frac{\alpha_0}{\chi_{T0}}\tau_{PV}\frac{dT}{dt}+\frac{\chi_{T0}}{\chi_{T\infty}}\tau_{TV}^2\frac{d^2p}{dt^2}+\frac{\tau_{TP}^2}{\upsilon_0\chi_{T0}}\frac{d^2\upsilon}{dt^2}-\frac{\alpha_0^2}{\chi_{T0}\alpha_\infty}\tau_{PV}^2\frac{d^2T}{dt^2}-$$

$$-\left(\frac{\chi_{T0}}{\chi_{T\infty}}\right)^2\tau_{TV}^3\frac{d^3p}{dt^3}-\frac{\tau_{TP}^3}{\upsilon_0\chi_{T0}}\frac{d^3\upsilon}{dt^3}+\frac{\alpha_0^3}{\chi_{T0}\alpha_\infty^2}\tau_{PV}^3\frac{d^3T}{dt^3}+ \qquad (96)$$

$$+\Gamma\varepsilon_r\left\{\tau_{TV}\left[-\frac{\upsilon}{\upsilon_x}p_{xxx}+p_{xx}-\frac{\upsilon_x}{\upsilon}p_x\right]+\frac{\tau_{TP}}{\upsilon_0\chi_{T0}}\left[-\frac{\upsilon}{\upsilon_x}\upsilon_{xxx}+\upsilon_{xx}-\frac{(\upsilon_x)^2}{\upsilon}\right]-\right.$$

$$-\frac{\alpha_0}{\chi_{\mathrm{T}0}}\tau_{\mathrm{PV}}\left[-\frac{v}{v_x}T_{xxx}+T_{xx}-\frac{v_x}{v}T_x\right]\Bigg\}.$$

Omitting the third time and spatial derivatives in dynamical equation of state (96) we pass to the dynamical equation of state for nonlocal media (66). To completely take into account the temperature terms in equation (96) we use relation (40) and assume that the processes studied are adiabatic ($\mathrm{S}=\mathrm{const}$, $\Gamma_{\mathrm{V}\infty}=\Gamma_{\mathrm{V}0}$). Then equation (96) takes the form

$$p-p_0=\frac{c_{S0}^2}{v_0\gamma_0}\left(1-\frac{v}{v_0}\right)+\frac{c_{S0}^2\alpha_0 T_0}{v_0\gamma_0}\left[\left(\frac{v}{v_0}\right)^{-\Gamma_{\mathrm{V}0}}-1\right]-\frac{\chi_{\mathrm{T}\infty}}{\chi_{\mathrm{T}0}}\tau_{\mathrm{TP}}\frac{dp}{dt}-\frac{\tau_{\mathrm{TP}}}{v_0\chi_{\mathrm{T}0}}(1+a)\frac{dv}{dt}+$$

$$+\frac{\chi_{\mathrm{T}\infty}}{\chi_{\mathrm{T}0}}\tau_{\mathrm{TP}}^2\frac{d^2p}{dt^2}+\frac{\tau_{\mathrm{TP}}^2}{v_0\chi_{\mathrm{T}0}}\left[(1+a)\frac{d^2v}{dt^2}-(\Gamma_{\mathrm{V}0}+1)\,a\frac{1}{v}\left(\frac{dv}{dt}\right)^2\right]-$$

$$-\frac{\chi_{\mathrm{T}\infty}}{\chi_{\mathrm{T}0}}\tau_{\mathrm{TP}}^3\frac{d^3p}{dt^3}-\frac{\tau_{\mathrm{TP}}^3}{v_0\chi_{\mathrm{T}0}}\left[(1+a)\frac{d^3v}{dt^3}-3(\Gamma_{\mathrm{V}0}+1)\,a\frac{1}{v}\frac{dv}{dt}\frac{d^2v}{dt^2}+\right.$$

$$\left.+(\Gamma_{\mathrm{V}0}+1)(\Gamma_{\mathrm{V}0}+2)\,a\frac{1}{v^2}\left(\frac{dv}{dt}\right)^3\right]+\Gamma\varepsilon_r\tau_{\mathrm{TP}}\left\{\frac{\chi_{\mathrm{T}\infty}}{\chi_{\mathrm{T}0}}\left[-\frac{v}{v_x}p_{xxx}+p_{xx}-\frac{v_x}{v}p_x\right]+\right.$$

$$\left.+\frac{1}{v_0\chi_{\mathrm{T}0}}\left[-\frac{v}{v_x}(1+a)\,v_{xxx}+(1+(3\Gamma_{\mathrm{V}0}+4)\,a)\,v_{xx}-\left(1+(\Gamma_{\mathrm{V}0}+2)^2\,a\right)\frac{(v_x)^2}{v}\right]\right\},$$

where the parameter $a$ is defined by expression (63).

# 15 Appendix

In order to change variables from $(p,\,v,\,T)$ to $(p,v,S)$ in equation (46) we must express $\frac{d^2T}{dt^2}$, $T_x$, $T_{xx}$ in the variables $S$ and $v$. Using formula (40), we get

$$\frac{dT}{dt}=\frac{T_0}{C_{\mathrm{V}0}}e^{\frac{S-S_0}{C_{\mathrm{V}\infty}}}v_0^{\Gamma_{\mathrm{V}\infty}}v^{-\Gamma_{\mathrm{V}\infty}}\frac{dS}{dt}-T_0e^{\frac{S-S_0}{C_{\mathrm{V}\infty}}}\Gamma_{\mathrm{V}\infty}v_0^{\Gamma_{\mathrm{V}\infty}}v_0^{-\Gamma_{\mathrm{V}\infty}-1}\frac{dv}{dt},\qquad(A1)$$

$$\frac{d^2T}{dt^2}=\frac{T_0}{C_{V\infty}^2}e^{\frac{S-S_0}{C_{V\infty}}}\upsilon_0^{\Gamma_{V\infty}}\upsilon^{-\Gamma_{V\infty}}\left(\frac{dS}{dt}\right)^2-\frac{T_0}{C_{V\infty}}e^{\frac{S-S_0}{C_{V\infty}}}\Gamma_{V\infty}\upsilon_0^{\Gamma_{V\infty}}\upsilon^{-\Gamma_{V\infty}-1}\frac{d\upsilon}{dt}\frac{dS}{dt}+$$
$$+\frac{T_0}{C_{V\infty}}e^{\frac{S-S_0}{C_{V\infty}}}\upsilon_0^{\Gamma_{V\infty}}\upsilon^{-\Gamma_{V\infty}}\frac{d^2S}{dt^2}-\frac{T_0}{C_{V\infty}}e^{\frac{S-S_0}{C_{V\infty}}}\Gamma_{V\infty}\upsilon_0^{\Gamma_{V\infty}}\upsilon^{-\Gamma_{V\infty}-1}\frac{dS}{dt}\frac{d\upsilon}{dt}+$$
$$+T_0e^{\frac{S-S_0}{C_{V\infty}}}\Gamma_{V\infty}\upsilon_0^{\Gamma_{V\infty}}\left(\Gamma_{V\infty}+1\right)\upsilon^{-\Gamma_{V\infty}-2}\left(\frac{d\upsilon}{dt}\right)^2-T_0e^{\frac{S-S_0}{C_{V\infty}}}\Gamma_{V\infty}\upsilon_0^{\Gamma_{V\infty}}\upsilon^{-\Gamma_{V\infty}-1}\frac{d^2\upsilon}{dt^2}. \quad (A2)$$

For adiabatic processes ($\sigma(S)=1$, $S=S_0$, $dS/dt=0$, $\Gamma_{V\infty}=\Gamma_{V0}$) $\frac{dT}{dt}=-T_0\Gamma_{V0}\upsilon_0^{\Gamma_{V0}}\upsilon_0^{-\Gamma_{V0}-1}\frac{d\upsilon}{dt}$,

$$\frac{d^2T}{dt^2}=-T_0\Gamma_{V0}\left(\Gamma_{V0}+1\right)\upsilon_0^{\Gamma_{V0}}\upsilon_0^{-\Gamma_{V0}-2}\left(\frac{d\upsilon}{dt}\right)^2-T_0\Gamma_{V0}\upsilon_0^{\Gamma_{V0}}\upsilon^{-\Gamma_{V0}-1}\frac{d^2\upsilon}{dt^2}. \quad (A3)$$

If $\upsilon=1/\rho$, then (A3) takes the form $\frac{dT}{dt}=-T_0\Gamma_{V0}\rho_0^{-\Gamma_{V0}}\rho^{\Gamma_{V0}-1}\frac{d\rho}{dt}$,

$$\frac{d^2T}{dt^2}=T_0\Gamma_{V0}\rho_0^{-\Gamma_{V0}}\rho^{\Gamma_{V0}-2}\left(\Gamma_{V0}-1\right)\left(\frac{d\rho}{dt}\right)^2+T_0\Gamma_{V0}\rho_0^{-\Gamma_{V0}}\rho^{\Gamma_{V0}-1}\frac{d^2\rho}{dt^2}, \quad (A4)$$

From formula (40) we find $T_x$:

$$T_x=\frac{T_0}{C_{V\infty}}e^{\frac{S-S_0}{C_{V\infty}}}\upsilon_0^{\Gamma_{V0}}\upsilon^{-\Gamma_{V0}}S_x-T_0e^{\frac{S-S_0}{C_{V\infty}}}\Gamma_{V0}\upsilon_0^{\Gamma_{V0}}\upsilon^{-\Gamma_{V0}-1}\upsilon_x. \quad (A5)$$

For adiabatic processes

$$T_x=-T_0\Gamma_{V0}\upsilon_0^{\Gamma_{V0}}\upsilon^{-\Gamma_{V0}-1}\upsilon_x, \quad (A6)$$

If we put $\upsilon=1/\rho$, then

$$T_x=T_0\Gamma_{V0}\rho_0^{-\Gamma_{V0}}\rho^{\Gamma_{V0}-1}\rho_x. \quad (A7)$$

Let us find $T_{xx}$ for the case of adiabatic processes (from (A6)):

$$T_{xx}=T_0\Gamma_{V0}\left(\Gamma_{V0}+1\right)\upsilon_0^{\Gamma_{V0}}\upsilon^{-\Gamma_{V0}-2}\left(\upsilon_x\right)^2-T_0\Gamma_{V0}\upsilon_0^{\Gamma_{V0}}\upsilon^{-\Gamma_{V0}-1}\upsilon_{xx}. \quad (A8)$$

Using $\upsilon=1/\rho$ we get the following expression

$$T_{xx}=T_0\Gamma_{V0}\rho_0^{-\Gamma_{V0}}\rho^{\Gamma_{V0}-2}\left(\Gamma_{V0}-1\right)\left(\rho_x\right)^2+T_0\Gamma_{V0}\rho_0^{-\Gamma_{V0}}\rho^{\Gamma_{V0}-1}\rho_{xx}. \quad (A9)$$

Combining formulae (A6) and (A9) we get

$$T_{xx} + \rho^{-1}\rho_x T_x = T_0\Gamma_{\mathrm{V}0}^2\rho_0^{-\Gamma_{\mathrm{V}0}}\rho^{\Gamma_{\mathrm{V}0}-2}\left(\rho_x\right)^2 + T_0\Gamma_{\mathrm{V}0}\rho_0^{-\Gamma_{\mathrm{V}0}}\rho^{\Gamma_{\mathrm{V}0}-1}\rho_{xx}. \quad (A11)$$

Making relations (A4), (A7), (A9) the following identities

$$\begin{aligned} &\upsilon = \rho^{-1}, \quad \upsilon_x = -\rho^{-2}\rho_x, \quad \upsilon_{xx} = 2\rho^{-3}\left(\rho_x\right)^2 - \rho^{-2}\rho_{xx}, \\ &\frac{d\upsilon}{dt} = -\rho^{-2}\frac{d\rho}{dt}, \quad \frac{d^2\upsilon}{dt^2} = 2\rho^{-3}\left(\frac{d\rho}{dt}\right)^2 - \rho^{-2}\frac{d^2\rho}{dt^2} \end{aligned} \quad (A10)$$

were used.

# References


[1] Anderson D., Tannehil D., Platcher P. Computational hydromechanics and heat, Moscow, Mir, 1986. – Vol. 1. – 384 p.

[2] Andronov A., Leontovich E., Gordon I., Maier A. Qualitative theory of second order dynamic systems, Moscow, Nauka Publ., 1976.

[3] Anischenko V.S. Complicated oscillations in common systems, Moscow, 1990.

[4] Arnold T.I. Geometrical methods in the theory of ordinary differential equations, New York, Heidelberg, Berlin, Springer-Verlag, 1982.

[5] Astarita G., Marucci G. Principles of non-newtonian fluid mechanics, McGraw-Hill, 1974.

[6] Balsa T.F. Amplitude equations for wave packets in slightly inhomogeneous unstable flows // J.Fluid Mech, 1989. – Vol.204.– PP.433-455.

[7] Barannyk A., Yurik I., Construction of exact solutions of diffusion equations, in *Proceedinds of Fifth International Conference "Symmetry in Nonlinear Mathematical Physics"* (June 23–29, 2003, Kyiv), Editors A.G. Nikitin and V.M. Boyko, Proceedings of Institute of Mathematics, Kyiv, 2004, Vol. 50, Part 1. – PP.29–33.

[8] Barenblatt G.I. Self-similarity and intermediate asymptotics, Gidrometeoizdat, Leningrad, 1978.

[9] Barenblatt G.I. Scaling, self-similarity and intermediate asymptotics, Cambridge Univ. Press, 1996.

[10] Bartucelli M., Conetantin P., Doering C.H. Hard turbulence in a finite dimensional dynamical system // Phys. Letters A., 1989. – Vol.142, No 6,7. – PP. 349-353.

[11] Bass J. Stationary functions and their applications to turbulence. 1.Stationary Functions // Journal of Mathematical Analysis and Applications, 1974. - Vol.47. - PP.354–399.

[12] Batchelor G.K. An introduction to fluid dynamics, Moscow, Mir, 1973. – 760p.

[13] Barakhtin B.K., Meshcheryakov Yu.I., Savenkov G.G. Microstructure of materials and the model for high-speed penetration of plane cumulative jets // Technical physics (ZhTF), 1991. – Vol.61, No.6. – PP.8–12.

[14] Baumann G. Symmetry analysis of differential equations with "Mathematica", New York, Springer, 2000.

[15] Bauer H. J. Phenomenological theory of the relaxation phenomena in gases, in *Physical Acoustics* (edited by W. P. Mason). – Mir: Moscow, 1968. – Vol. 2, issue A. – 300 p.

[16] Berezhnev I.A., Nikolaev A.V., On the propagation of strong nonlinear pulse in blocked-hierarchic media // Phys. Earth Planetary Int., 1988. – Vol.50. – PP.83–87.

[17] Berje P., Pomo I., Vidal K. The order in a chaos. On deterministic approach to turbulency / transl. from French, Moscow, Mir, 1991.

[18] Biler P., Karch G. (eds.) Self-similar solutions in nonlinear PDEs, Banach Center Publications, **74**, Warsaw, 2006.

[19] Bisyarin M.A. Distribution of non-adiabatic perturbations in a relaxing environment // Physics of combustion and explosion, 1987. – Vol.3. – PP.95–99.

[20] Bluman G., and Cole J.D. Similarity methods for differential equations, New York, Springer, 1974.

[21] Bluman G., Kumei S. Symmetries and differential equations, New York, Springer, 1989.

[22] Bluman G. Symmetry, integrability and geometry: methods and applications, 2005. – **1**, Paper 011. – 16 p. (http://www.emis.de/ journals/SIGMA/2005/Paper011/index.html).

[23] Bogaevsky V., Povzner A. On one-dimensional shock waves // Int.J.Non-linear Mechanics, 1979. – Vol.13. – PP.337-349.

[24] Bogolyubov N. H. The problems of dynamical theory in statistical physics. – M.: Gosteohizdat, 1947. – 487 p.

[25] Bogolyubov N. N., Mitropolskyy Yu.A. Asimptotical methods in the theory of nonlinear oscilations. – Moscow: Nauka, 1974. – 504 p.

[26] Borisov A.A. The equation for the structure of high-frequency perturbations with finite amplitude in a medium with internal processes.Thermal studies. (edited by S. S. Kutateladze). – Novosibirsk, 1977. - PP. 52-55.

[27] Boiti M., Leon J., Penpinelli F. Waves in the Davey-Stewartson equation // Inverse Problems, 1991. – Vol. 7, No.2, April. – PP.175.

[28] Boldrighini C. and Franceschini N. Five-dimensional truncation of the plane incompressible Navier-Stokes equations // Comun. Math. Phys., 1979. – Vol.64, No.2. – PP.159-170.

[29] Book D.L., Boris J.P., Hain K. Flux - corrected transport II: Generalizations of the method // J. of Comput. Phys., 1975. – Vol. 18. – PP.248 – 283.

[30] Bountis T., Segur E. Logarithmic singularities and chaotic behhaviour in haroiltonian system // in *Math. Hydrodynam. and Integrability Dyn. Syst*, Workshop. La Jolla, 1981. – N.Y. 1982. – PP.279–292.

[31] Brenner P., Thomee V. Stability and Convergence Rates in Lp for Certain Difference Schemes // Mathematica Scandinavica, 1970. – Vol.27. – PP.5–23.

[32] Brushlinskaya N.N. Of the emergence of the periodic flow and the tori by laminar flow / in *Some questions of the rook mechanics*, 1968. – PP.57–79 (In Russian).

[33] Bullough R.K., Caudrey P.J (editors) Solitons, Berlin - Hew York, Springer, 1980.

[34] Burczlaff J., Zakrzewski W. $CP^2$ soliton scattering: the collective coordinate approximation // Nonlinearity,1998. – Vol.11, No.5, – PP.1311-1320.

[35] Burgers J.,M. The Nonlinear Diffusion Equation, Reiedl, Dordrecht, 1974.

[36] Calogero F., Degasperis S. Spectral transform and solitons, Vol. I, North-Holland Publishing Co., Amsterdam, New York, 1982.

[37] Cattaneo C., A form of heat conduction which eliminates the paradox of instantaneous propagation // Compte Rendus,1958. – Vol. 247. – PP.431–433.

[38] Cherniha R. New non-Lie ansatze and exact solutions of nonlinear rection-diffusion-convection equations // J. Phys. A: Math and Gen.,1998. – Vol. 31. – PP.81798198.

[39] Clarkson P., Kruskal M. New similarity reductions of the Boussinesq equation // J. Math. Phys., 1989. – Vol. 30. – PP.2201–2213.

[40] Clarksonz P., Mansfield L. Symmetry reductions and exact solutions of a class of nonlinear heat equations // Physica D, 1994. – Vol.70, No.3. – PP.250–288.

[41] Cornille H., Gervois A. Bi-solitons for a class of non-linear equations with quadratic non-linearity. I // Physica D, 1982. – Vol.6, No.1. – PP.1–28.

[42] Cornille H., Gervois A. Connection between the existence of bisolitons for quadratic nonlinearities and a factorization of the associate linear operator // J. Math. Phys., 1983. – Vol.24, No.8. – PP. 2042-2055.

[43] Danevych T.B., Danylenko V.A. Investigations of the influence of spatial nonlocality on the waves spreading in a relaxing medium // S.I.Subbotin institute of geophusics NAS of Ukraine, preprint. – Kyiv, 1994. – 22 p.

[44] Danevych T.B., Danylenko V.A. Influence of spatial nonlocality on the waves spreading in a relaxing medium // Dop. NAS of Ukraine. – 1995. – Vol.11. – P. 83-86.

[45] Danevych T.B., Danylenko V.A. The equation of state of nonlinear medium with internal variables taking into account spatial and temporal nonlocalities // Dop. NAS of Ukraine . - 1998. - No.10. - P. 133 - 136.

[46] Danevych T.B., Danylenko V.A. Governing equation of nonlinear medium internal variables taking into account spatial and temporal nonlocalities: Prepr. / NAS of Ukraine. S.I. Subbotin inst. of geophysics. - K.: 1999. - 22 p.

[47] Danevych T.B., Danylenko V.A. Exact analitical solutions of nonlinear equations for dynamics of relaxing media with spatial and temporal nonlocalities. // Dop. NAS of Ukraine . - 2004. - No.3. - P. 110 - 114.

[48] Danevych T.B., Danylenko V.A. Exact analitical solutions of nonlinear equations for dynamics of nonlocal media taking into account of temperature terms and the nonstationarity of generalized thermodynamical force. // Dop. NAS of Ukraine . - 2004. - No.4. - P. 106 - 112.

[49] Danevych T. B., Danylenko V.A. Nonlinear nonlocal models of multicomponent relaxing media with interior oscillators // Dop. NAS of Ukraine. - 2005. - Vol.1. - PP.106-110 (Ukr.).

[50] Danevych T.B,, Danylenko V,A., Korolevich V.J., Sorokina V.Y., Vladimirov V.A. The equations of hydrodynamics for active media. Symmetries, qualitative analysis and wave structures evolution

[51] Danilov V.G., Maslov V.P. and Volosov K.A. Mathematical Modelling of Heat and Mass Transfer Processes, Kluver, Dordrecht 1995.

[52] Danylenko V.A. Nonlinear models of multicomponent relaxing media / in *Modeling of dynamics of deformable media*, Kiyv, Nauk.dumka, 1993.

[53] Danylenko V.A., Christenyuk V.O., Korolevich V.Yu. and Makarenko A.S.

[54] Danylenko V.A., Christenyuk V.O. On the wave structures formation at the high-rate pulse load application to the relaxing geodynamical media // Reports NAS of Ukraine, 1994. – No.11. – P.112-115.

[55] Danylenko V.A., Kudinov V.M., Makarenko A.S. The influence of the memory effects for the dissipative structure creations under the high-rate processes. – Kiev, Ukraine. Preprint, IES - 83 - 1, 1983. – 60 p. (in Russian).

[56] Danylenko V.A., Kudinov V.H., Makarenko A. S. The influence of the memory effects for the burning and explosion processes // Phys. of burning and explosion, 1984 – Vol. 20, No.3. – PP. (In Russian). Translated in Journ. Eng. Phys., Consultant Bureau, NY, 1984, Vol.47, No.5.

[57] Danylenko V.A., Kudinov V.M., Makarenko A.S., // Inzh. Fiz. Zhurnal, **47**, 843 (1985).

[58] Danylenko V.A., Skurativskiy S. I. Invariant periodic and chaotic solutions of a model of multicomponent medium with physical nonlinearity, spatial and temporal nonlocalities // Dop. NAS of Ukraine, 2005. – No. 8. – P. 112–115.

[59] Danylenko V.A., Skurativskiy S. I. Chaotic invariant solutions of nonlinear nonlocal models for multicomponent media with internel oscilators // Dop. NAS of Ukraine, 2006. – Vol.9. – P. 111–115.

[60] Danylenko V.A., Skurativskyy S. I. Invariant chaotic and quasi-periodic solutions of nonlinear nonlocal models of relaxing media // Reports of mathematical physics, 2007. – Vol.59, No. 1. – PP. 45–51.

[61] Danylenko V.A., Sorokina V.V., Vladimirov V.A. On the Governing Equations in Relaxing Media Models and Self-Similar Quasiperiodoc Solutions // Journal of Physics A, 1993. – Vol. 26. – PP.7125–7135.

[62] Danylenko V.A. and Vladimirov V.A. // in *Boundary problems of the mathematical physics*, Kyiv, Nauk.dumka, 1990.

[63] Danylenko V.A., Vladimirov V.A. Qualitative and numerical study of the non-equilibrium high-rate processes in relaxing media // Control and Cybernetics,1996. – Vol.25, No.3 – PP.569–581.

[64] Danylenko V.A., Vladimirov V.A. On the self-similar solutions of generalized hydrodynamics equations and non-linear wave patterns // J. Nonlin. Math. Phys., 1997. – Vol.4, No. 1–2. – PP.25–32.

[65] Danylenko V.A., Vladimirov Y.A., Korolevich V.Yu. The nonlinear models of multicomponent relaxing media. Dynamic of wave structures and qualitary analysis. Part 1, Preprint. Ukr. Geophys. Inat., Kiev, 1990. – 40 p. (in Russian).

[66] Danylenko V.A., Vladimirov V.A., Korolevich V.Yu. Nonlinear models of multi-component relaxing media // Dopovidi Acad. Sci. Ukraine, 1992. – No. 1 – PP. 89–93.

[67] Danylenko V.A., Vladimirov V.A., Skurativskiy S. I. Quasi-periodic and chaotic solutions of a model of multicomponent medium // Dop. NAS of Ukraine, 2004. – No. 12. – PP. 104–108.

[68] Danylenko V.A., Kudinov V.M., Makarenko A.S. The ignition and burning of despersive systems // Doklady of Ukr.A.S., Ser.A., 1985. – No. 5. – PP. 69–72.

[69] Davydov A.S. Solitons and energy transfer along protein molecules, // Journ. of theoretical biology, 1977. – Vol. 66. – PP. 379-387.

[70] Day W. A. The thermodynamics of simple materials with fading memory, Springer-Verlag, Berlin, 1972.

[71] De Groot S.R., Mazur P. Nonequilibrium Thermodynamics, North-Holland, Amsterdam, 1962.

[72] Demekhin Ye.A., Shkadov V.Ya. On the solitons in dissipative media, in *Hydrodynamics and thermal-mass transfer*, Editor G.I. Kutateladze, Novosibirsk, Insitute of Thermal Physics, 1985. – PP.32–48.

[73] Demekhin Y.A., Tokarev G. Yu and Shkadov V. Ya. Hierarchy of bifurcations of space-periodic structures in nonlinear model of active dissipative media // Physica D, 1991. – Vol. 52. – PP. 338–361.

[74] DiPerna R.J. Convergence of approximate solutons to conservation laws // Aroh.Rational Mech.Anal., 1983. - Vol.82. - PP.17-70.

[75] DiPerna R.J. Convergence of the viscosity method for isentropic gas dynamics // Communications in Mathematical Physics, 1983. – vol.91, No.1. – PP.1–30.

[76] DiPerna R.J. Mesuare-valued solutions to conservation laws // Arch. Ration. Mech. Anal., 1985. – Vol.86. – PP.223–270.

[77] Dodd R.K., Eilbek J.C., Gibbon J.D., Morris H.C. Solitons and Nonlinear Wave Equations, Academic Press, London, 1984.

[78] Dubois D. Introduction to Computing Anticipatory Systems // International Journal of Computing Anticipatory Systems, 1998. – Vol.2, – PP.3–14.

[79] Dubois D. Derivation of Quantum relativist Electromagnetic Field From Space-Time Shifts: Connection to the Inertial Mass in the Zero Point Field. // Procceed. of the Fifth European Meeting on Cybernetics and System Research, ed. Robert Trapple., 2000. – Vol. 1. – PP.170–175.

[80] Dubois D. New trends in Computing Anticipatory Systems: Emergence of Artifial Conscious Intelligence with Machine Learning Natural Language // Computing Anticipatory Systems. Proceed. CASYS'07 – Eight Int. Conf. Liege, Belgium, 6011 August 2007, ed. Daniel M. Dubois, AIP Conf. Proceedings 1051, N.Y., 2008.

[81] Egorov Yu. V. A contribution to the theory of generalized functions //Russian Mathematical Surveys (UMN), 1990. – Vol.45:5(275). – PP.3–40.

[82] Faddeev L.D., Takhtajan L.A. Hamiltonian methods in the theory of solitons, Berlin, Springer, 1987.

[83] Fahmy E.S., Abdusalam h.A., Raslan K.R., On the solutions of the time delayed Burgers equation // Nonlinear Analysis, 2008. – Vol. 69, – PP.4475–4786.

[84] Fan E. Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method // Journ. Phys. A: Math. Gen., 2002. – Vol. 35, No.32. – PP.6853–6872.

[85] Ficket W., Davis W.C., Detonation, Berkley, Univ. of California Press, 1979.

[86] Fokas A.S., Ablowitz M.J. On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane // J.Math.Phys., 1984. – Vol.25, No.8. – PP.2494-2505.

[87] Fokas A.S., Liu Q.M. Nonlinear interaction of traveling waves of nonintegrable equations // Phys. Rev. Letters,1994. – Vol. 72. – PP.3293–3296.

[88] Fowler A.C., Gibbon S.D., McGuinness M.J. The Complex Lorenz equation. // Physica D:Nonlinear Phenomena, 1982. – Vol. 4, No.2 – PP.139–163.

[89] Franceschini N., Tebaldy C. Seven-mode truncation of the plane incompressible Navier-Stokes equations // J. of Statist.Phys., 1981. – Vol.25, No.3. – PP.397–408.

[90] Frank-Kamenetski D.A. Diffusion and heattransfer in chemical kinetics, Moscow, Nauka, 1967. – 492 p.

[91] Fushchych W., Shtelen W., Serov N. Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluver Publishers, Dordrecht, 1993.

[92] Fushchich W., Tsyfra I. On a reduction and solutions of nonlinear wave equations with broken symmetry // J. Phys. A: Math. Gen., 1987. – Vol.20, No.2, – L45.

[93] Galaktionov V.A. et.al. Heat localization in nonlinear media // Differential equations, 1981. – Vol.17, No.10. – PP.1826–1841.

[94] Galaktionov V.A. On conditions for the absence of global solutions of a class of quasi-linear parabolic equations // Computational Mathematics and Mathematical Physics, 1982. – Vol.22, No.2. – PP. 322–338.

[95] Galaktionov V.A., Samarskii A.A. On difference solutions of a class of quasilinear parabolic equations. Part I. // Computational Mathematics and Mathematical Physics, 1983. – Vol.23, No.3. – PP.646–659.

[96] Galaktionov V., Vazquez J. A stability technique for evolutionary PDEs: a dynamics systems approach, Birkhauser, 2004.

[97] Gallacher R. The theory of information and reliable connection, Moscow, Sov. radio, 1974. – 720 p.

[98] Gavrilyuk I.P. et al. Mesh scheme with exact spectrum for the quasilinear equation of heat conduction. - Preprint I.P.Matem. Akad. Nauk SSSR, Moscow, No.97, 1979.

[99] Gavrilyuk S.L., Shurgin S.M. Media with equations of state depending on derivatives // Appl. mech. and tech. phys.,1996. – Vol.37, No.2. – PP.35–49.

[100] Gerdt V. Gröbner bases and involutive methods for algebraic and differential equations // Mathematical and Computer modelling, 1997. – Vol.25, No.8-9. – PP.75–90.

[101] Glansdorf P., Prigogine I. Thermodynamics of structure, stability and fluctuations, New York, Wiley, 1971.

[102] Glassey R.T. On the blowing up of solutions to the Cauchy problem for non-linear Schrodinger equation // J. Math. Phys., 1977. – Vol.18, No.9. – PP.1794–1797.

[103] Glassey R.T. Blow-up theorems for nonlinear wave equations // Mathematische Zeitschrift, 1973. – Vol.132, No.3. – PP.183–203.

On the blowing up of solutions to the Cauchy problem for nonlinear Schrodinger equation // J. Math. Phys., 1977. – Vol.18, No.9. – PP.1794–1797.

[104] Glimm J. Solutions in large for nonlinear hyperbolic systems of equations // Comrn. Pure Appl. Math., 1965. – Vol. 18. – PP. 6970715.

[105] Godunov S. K. A Difference Scheme for Numerical Solution of Discontinuos Solution of Hydrodynamic Equations // Math. Sbornik, 47 (1959), 271-306, translated US Joint Publ. Res. Service, JPRS 7226, 1969.

[106] Gonchenko S., Turaev D., Šilnikov L. On models with non-rough Poincaré homoclinic curves // Physica D, 1993. – Vol.62, No.1-4. – PP.1–14.

[107] Goppa V.D. The information of words // Probl. peredachi inform., 1978. – Vol.14, No.3. – PP.3–18.(In Russian).

[108] Goppa V.D. Codes and information // Uspehi matematicheskih nauk,1984. – Vol.39, No.1. – PP. 77–120.

[109] Goulett P., Elphio R. Nature of spatial chaos // Phys. Rev.Let., 1987. – Vol. 58, No.5. – PP.431–434.

[110] Grebogi C.,Ott E., Yorke J.A. Chaotic attractors in crisis // Phys. Rev. Letters, 1982. – Vol.48. – P.1507.

[111] Grebogi C.,Ott E., Yorke J.A. Rises, sudden changes in chaotic attractors and transients to chaos // Physica D, 1983. – Vol.7. – P.181.

[112] Grechanyy O.A. The statistical theory of irreversible processes, Kyiv, Naukova dumka, 1989. - 376 p.

[113] Grunland A.V., Tuszynski J.A. Multivalued solutions of the $f4$-field equation transition to ergodicit and bifurcations // Phys. Lett., 1988. – Vol.133, No 6. – PP.298–304.

[114] Guckenheimer J., Holmes P. Nonlinear oscillations, dynamical systems and bifurcations of vector fields, Springer–Verlag: New York Inc, 1987.

[115] Gurtin R.M., Pipkin A.S. A genral theory of heat conduction with finite wave speeds // Arch. Rat. Mech. Anal.,1968. – Vol.31, No. 3. – PP.113–126.

[116] Hairer E., Norsett S.P., Wanner G. Solving ordinary differential equations.I. Nonstiff problems, Springer-Verlag, 2000. - 528 p.

[117] Haken H. Synergetics, Springer-Verlag, Berlin-Heideeberg, New-York, 1978. – 404 p.

[118] Halperin B.I., Kohlenberg P.O. Hydrodynamic theory of spin waves // Phys. Rev., 1969. – Vol.188, No. 2. – PP.898–919.

[119] Halperin B.I., Saslow H.M. Hydrodynamic theory of spin waves in spin glass and other systems with nonlinear spin orientations. // Phys. Rev. B, 1977. – Vol. 16, No.5. – PP.2154–2162.

[120] Hargittai I., Hargittai M. Symmetry through the eyes of a chemist, New York, Plenum Press, 1995.

[121] Hartley R.V.L. Transmission of information. // Bell.Syst.Tech.J.,1928. – Vol.7. – PP.535–563.

[122] Hassard B.D., Kazarinoff N.D., Wan Y.-H., Theory and Applications of the Hopf Bifurcation, Springer, New York, 1981.

[123] Hayes W.D., Energy invariant for geometric acoustics in a moving media // Phys. Fluids., 1968. – Vol.11. – PP. 1654–1656.

[124] Hopfield J. T. Neural networkes and physical systems with evergent collective computation abilities // Proc. NAS of USA, 1982. – Vol.79, No.8. – PP.2554–2558.

[125] Ibragimov N.H. Groups of transformations in mathematical physics, Nauka, Moscow, 1983.

[126] Isachenko M.B., Yan'kov V.V. Generation of high harmonics of laser radiation in plasma during electron wavebreaking // Journal of Experimental and Theoretical Physics, 1984. – Vol.87, No. 6(12). – PP.1914–1925.

[127] Iskol'dskii A. M. and Romenskii E. I., A dynamic model of a thermoelastic continuous medium with pressure relaxation // Journ. Appl. Mech. and Tech. Physics (PMTF),1984. – Vol.25, No. 2. – PP.286–291.

[128] Ivanenko V.I., Melnik V.S. On the synthesis of attracting sets for the system with distributed parameters // Avtomatika, 1993. – No.1. – PP.55–73.

[129] Jieudonneand J., Carrell J.B. Invariant theory old and new, New York, Academic Press, 1971.

[130] Joseph D.D., Preziozi L. Heat Waves // Review of Modern Physics, 1989. – Vol. 61, No. 1. – PP.41–73.

[131] Johnson P.A., Rasolofosaon P.N.J. Manifestation of nonlinear elastisity in rock: convincing evidence over large frequency and strain intervals from laboratory studies // Nonlinear Processes in Geophysics, 1996. – No. 3. – PP.77–88.

[132] Jou D., Perez-Garcia C.,Garcia-Colin L.S., Lopez de Haro M., Rodriguez R. F. Generalized hydrodynamics and extended irreversible thermodynamics // Phys. Rev. A, 1985. – Vol.31, No.4. – PP.2502–2508.

[133] Jumaire Q. Soure approaches to the meajure of the amaunt of information involved by a form // Syst. Anal. Model. Simul., 1986. – Vol.3, No.6. – PP.479–506.

[134] Kadomtsev V. V. Collective phenomena in plasma, Moscow, Nauka, 1976. - 231 p.

[135] Kadomtsev V. V., Mikhaylovskiy A.B., Timofeev A.V. Waves of negative energy in dispersing media // Journal of Experimental and Theoretical Physics (JETF), 1964. – Vol.17, No.6(12). – PP.2266–2268.

[136] Kaplan S. On the growth of solutions of quasilinear parabolic equations // Communs. Pure and Appl. Math., 1963. – Vol.16, No.3. – PP.305–330.

[137] Kar S., Banik S.K., Ray Sh. Exact solutions of fisher and Burgers equations with finite transport memory // J. Phys. A: Math. Theor., 2003. – Vol. 36, No.11. – PP.2771–2780. (see also arXiv:nlin/0112025, 2002)

[138] Karpman V.I. Nonlinear waves in dispersive media, Moscow, Nauka, 1973.

[139] Kawahara T., Tanaka M. Interactions of traveling fronts: an exact solution of a nonlinear diffusion equations // Phys. Lett., 1983. – Vol. 97. – PP. 311.

[140] Kestenboym Kh.S., Roslyakov G.S. Chudov A.A. The point explosion. The methods of calculations. The tables, Moscow, Nauka, 1974.

[141] Kevorkian J., Cole J.D. Perturbation methods in applied mathematicsm, Springer, New York, 1981. - 300 p.

[142] Khonkin A.D. On the nonequelibrium fluctuations influence to the apperence of turbulence / in *Turbulent flows*, Nauka, Moscow, 1974. – PP.141–143.

[143] Khristov Kh.I., Nartov V.P. Point random functions and large-scale turbulance, Novosibirsk, Nauka, 1992. – 160 p.

[144] Klimontovich Yu. L. Statistical physics, Moscow, Nauka, 1982. – 608 p.

[145] Klimontovich Yu. L. Turbulance motion and the structure of chaos, Moscow, Mauka, 1990. – 320 p.

[146] Koga T. An introduction in kinetic theory of stochastic gas processes, Moscow, Nauka, 1983. – 272 p.

[147] Kolesnik T.V., Dzhuraev I.N. Application of exact difference schemes to estimating the rate of convergence of the lattice method for second order hyperbolic equations // J. Numer. Appl. Math. (Kyiv National University), 1983. – No.50. – PP.55-59.

[148] Korn G.A., Korn T.M. Mathematical handbook, New York, McGraw-Hill, 1968.

[149] Korniak V., Fushchych W. Computer algebra application for determining Lie and Lie-Backlund symmetries in differential equations // J. Symb. Comp., 1989. – Vol. 7(6). – P.611.

[150] Kraichnan R. H. Stochastic modeling of isotropic turbulence / in *New trends in turbulence*, Springer-Verlag, 1991. – PP.1– 54.

[151] Krasnev Yu.K. Evolution of "tornados" / in *Nonlinear waves. Structures and bifurcations*, Moscow, Nauka, 1987. – PP.174–189.

[152] Krasny R. Computing vortex sheet motion // Proc. Int. Congress of Math., Kyoto, Japan, 1990. – PP.1573–1583.

[153] Krasnoselskiy V.V. About nonlinear motions of plasma broadwise magnetic polya // GETF, 1985. – Vol.89, No 2(89). – PP.498–510.

[154] Kriviy I. Anticipatory systems in population dynamics // Int. J. of Computing Anticipatory Systems, – 2008. – Vol.21. – PP.121–131.

[155] Kudryashov N.A. Backlund transformations for fourth order partial differential equations with Burgers-KdV nonlinearity // Doklady of AS USSR, 1988. – Vol. 300, No.2. – PP.342–345.

[156] Kudryashov N.A. Exact solutions of $N$-order equations with Burgers-KdV nonlinearity // Matematicheskoje modelirovanije, 1989. – Vol.1, No.6. – P. 57–65.

[157] Kudryashov N.A. Exact solution of the generelized Kuramoto-Sivashinsky equation // Physics Letters A., 1990. – Vol. 147, No.5,6. – PP.287–291.

[158] Kuleshov V.A., Makarenko A.S. On the dispersive turbulance in a some descrete and distributed systems. Type-scrop registrated in AISHCI, Moscow, No.7491, 1985, – 44 p. (in Russian). Resume in J. Engrg. Phys. Consultant Bureau, NY, 1985. – Vol.48, No.4 – P.664.

[159] Kurdyumov S.P. Evolution and self-organization law in complex systems // Inter. Journal of Morden Physics. C, 1990. – Vol.1, No.4. – PP.299–327.

[160] Kutafina E.V. Pewne typy rozwiazan ewolucyjnych modeli noeliniowych. – Ph. D. thesis, Faculty of Applied Mathematics, Academy of Science of Technology, Krakow, 2008.

[161] Kuznetsov Yu.A. One-dimensional separatrises of the system of ODE, depending on parameters, Puschino, 1983. – 46 p.

[162] Ladyzhenskaya O.A. The mathematical quations of viscous incompressible fluid dynamics, Moscow, Nauka, 1970. – 288 p.

[163] Ladyzhenskaya O. A. On the determination of minimal global attractors for the Navier–Stokes and other partial differential equations // Russian Mathematical Surveys (UMN), 1987. – Vol.42:6(258). – PP.25–60.

[164] Lam J.L. Introduction to solitons theory, Moscow, Mir, 1983. - 294 p.

[165] Lamd J. Thermal relaxation in liquids / in *Physical acoustics* (edited by W. P. Mason), Moscow, Mir, 1968. – Vol. 2, Part A.– 300 p.

[166] Landau L.D., Lifshitz E.M. Hydrodynamics, Nauka, Moscow, 1986. - 736 p.

[167] Larkin N.A., Novikov V.A., Yanenko N.N. The nonlinear equations of changable type, Novosibirsk, Nauka, 1983. – 270 p.

[168] Lasota A. Stable and chaotic solutions of the first order partial differential equations // Nonlinear Anal.Theory Math. Appl., 1981. – Vol.5, No.1. – PP.1181-1183.

[169] Lazaridi A.N., Nesterenko V.F. Obnaruzhenije uedinennih woln novogo tipa v odnomernoy zernistoy srede // Appl. mech. and tech. phys., 1985. – No. 3. – PP.115–118.

[170] Lax P.D. Shock waves and entropy // In: *Contribution to Nonlinear Functional Analysis* (E.A.Zarantello, ed), Academic Press, 1971. – PP.603–634.

[171] Lesier M. Turbulence in fluid. Stochastic and numerical modeling, Martin Nijhoff Publishers, 1987. – 286 p.

[172] Leontiev A.F. Tselyje funktsii. Riady eksponent, Moscow, Nauka Publ., 1983.

[173] Levine H. A. Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: The method of unbounded fourier coefficients // Mathematische Annalen., 1975. – Vol. 214, No.3. – PP.205–220.

[174] Leyton M. A generative theory of shape, Berlin, Springer-Verlag, 2001.

[175] Li Y.A., Olver P.J. Convergence of solitary-wave solutions in a perturbed bi-hamiltonian dynamical system. 1. Compactons and peakons, http://www.math.umn.edu/ olver

[176] Li Y.A., Olver P.J., Rosenau Ph. Non-analtic solutions of nonlinear wave models / in *Nonlinear theory of generalized functions*, M. Grosser, G. Hormann, M. Kunzinger and M. Oberguggenberger, eds., Research Notes in Mathematics, vol. 401, Chapmann and hall/CRC, New York, 1999. – PP.129–145 (see also http://www.math.umn.edu/ olver).

[177] Liboff R.L. Introduction to the theory of kinetic equations, Willey, Ney York, 1969. – 371 p.

[178] Lighthill J. Waves in fluids, Cambridge University Press, New York, 2005.

[179] Loskutov A.Yu., Mikhailov A.S. An introduction to synergetics, Moscow, Nauka, 1990.

[180] Loyciansky L.G. // Izv. Akad. Nauk USSR Ser. Fluid Mech. 2 (1982) 5.

[181] Luikov A.V. Application of irreversible thermodynamics methods to investigation of heat and mass transfer // Inzhenerno-Fiziceskii Zhurnal, 1965. – Vol.IX, No.3. – PP.287–304. (Translation in International Journal of Heat and Mass Transfer, 1966. – Vol.9, No.2. – PP.139–152.)

[182] Lyakhov G.M. Fundamentals of explosive waves dynamics in grounds and rock, Moscow, Nedra, 1974. – 192p.

[183] Lyakhov G.M. Waves in grounds and porous multicomponent media, Moscow, Nauka Publ., 1982. – 286p.

[184] Majda A.J. The interaction of Nonlinear Analysis and Modern Applied Mathematics // Proc. of Int. Congr. of Math., Kyoto, Japan, 1990. – PP.175–191.

[185] Makarenko A.S. Analysis of dispersion of difference schemes for the Klein - Gordon equation // Chisl. Met. Mech. Sploshn. Sredy, 1982. – Vol 13, No.3. – PP.81–90.

[186] Makarenko A.S. Of the special features of the heat and mass convection description in high-rate processes // Type-scrip. registrated in AISEII, II., N 6882-B 87, 1987. – 40 p. (in Russian). Resume: Jorn. of Engineer Physics. Consultants Bureau, NY, 1988. – Vol.54, No.2.

[187] Makarenko A.S. On one order measure of blow-up solutions // Memoir saved in Ukraine National Int. Technical. Inf., No.1504. UNTII,Kiev, 1988. – 86p.

[188] Makarenko A.S. Of the complicity of spatial objects // Sypt-scrip regietrated in Ukr.SIISTI, Kiev, 1988. – Vol.1504. – 86p.

[189] Makarenko A.S. On the complexity of space objects. // Memoir saved in Ukraine National Int. Technical. Inf., No. 1717-Yk89. UNTII, Kiev, 1989 – 46p.

[190] Makarenko A.S. Of the one blow-up regime odering mesuare // Typescrip registrated in Ukr-ISEPI. Kiev, No. 1504, 1988, – 86 p. (In Russian).

[191] Makarenko A.S. The mathematical modeling of hydrodynamical problems taking into account memory effects // Doklady NAS of Ukraine, 1994. – No.2. – PP.85–88.

[192] Makarenko: A.S., Mathematical modelling of memory effects influence on fast hydrodynamic and heat conduction processes // Control and Cybernetics, 1996. – Vol.25, No. 3. – PP.621–630.

[193] Makarenko A.S. New differential equation model for hydrodynamics with memry effects // Reports on Mathematical Physics, 2000. – Vol. 46, No.1/2. – PP.183–190.

[194] Makarenko A.S. Mathematical modelling of memory effects influence on fast hydrodynamic and heat conduction processes // Control and Cybernetics, 1996. – Vol. 25, No. 3. – PP.621–630.

[195] Makarenko A.S., Moskalkov M., Levkov S. On blow-up solutions in turbulence // Phys. Lett. A, 1997. – Vol. 23. – PP.391–397.

[196] Makarenko A.S. // Proceedings of the fifth Annual Seminar "NPCS'96", Vol. 8, Minsk, 1997.

[197] Makarenko A.S. Complexity of individual objects without including probability concept. Asymmetry, observer and perception problem. // Computing anticipatory systems: CASYS 2000 – Forth international conference. Edited by D. Dubois, American inst. of physics, AIP conference proceeding 573, pp 201 – P. 215.

[198] Makarenko A. Cellular Automata with Anticipation. Some new Research Problems // Abstract Book Int. Conf. CASYS 07, Liege, August, 2007.

[199] Makarenko A.S., Moskalkov M.N. One approach to numerical oscillations reducing // Computational and applied mathematics, Kiev, 1980. – No.41. – PP.65–71.

[200] Makarenko A.S., Moskalkov M. N. On dispersion of locally homogeneous schemes for the transport equation // Chisl. Met. Mech. Sploshn. Sredy, 1981. – Vol.12, No.2. – PP.64–70.

[201] Makarenko A.S., Moskalkov M.N. On the precision and dispertion of difference schemes // Zhurnal vich. metodov i math. fiziki, 1983. – Vol.23, No.4. – PP.999–1004.

[202] Makarenko A.S., Moskalkov M.N. Some solutions of model exuations or relaxed media // Preprint, Institute of Geophysics, Kiev, 1992. – 27p.

[203] Makarenko A.S., Moskalkov M.N. Some solution of relaxed media model equations // KPI, Kyiv, 1992.

[204] Makarenko A.S., Sagaidak L. Complexity calculations of geometric objects by symmetry analysys in physics, biology and signal processing // Proc. of the XV-th Int. Hutsulian Workshop "Category theory and applied systems analysis in decision-making problems" (april,2004, Kyiv; september, 2005, Sevasopol, Ukraine), Kyiv, 2006. – PP.163–172.

[205] Makarenko A., Stashenko A. Some two- steps discrete-time anticipatory models with 'boiling' multivaluedness // AIP Conference Proceedings edited by Daniel M. Dubois, USA, 2006. – Vol.839. – PP.265–272.

[206] Makarenko A.S., Swirsky T. Modeling self-origin of collapses in hydrodynamics with memory. // Proc. of the XV-th Int. Hutsulian Workshop "Category theory and applied systems analysis in decision-making problems" (april,2004, Kyiv; september, 2005, Sevasopol, Ukraine), Kyiv, 2006. – PP.183–206.

[207] Makarenko A.S. Toward the analysis of numerical schemes dispersion for Klein-Gordon equation // Numerical methods for continuum mechanics (Novosibirsk), 1982. – Vol.13, No.3. – PP.81–90.

[208] Maystrenko Yu.L., Stukanev I.I. The first boundary problem for the Burgers equations // Dynamical system and nonlinear phenomena, Kyiv, Institute of Math. NAS of Ukraine, 1990. – PP.26–38.

[209] Mishaikov K., Zgliczynski P., Rigorous numerics for PDEs:the Kuramoto-Sivashinski equation // Foundations of computional mathematics, 2001. – Vol.1. – PP.255–288.

[210] Maslov V.P. Breaking of the causality principle for unsteady equations of two- and three-dimensional gas dynamics at enough large Reynolds' numbers // Theoretical and Mathematical Physics, 1986. – Vol.69, No.3. – PP.361–378.

[211] Maslov V.P. Resonance non-correctness / in *Sovremennyje problemy prikladnoj matematiki i matematicheskoy fiziki*, Moscow, Nauka, 1988. – PP.60–62.

[212] Mel'nik V. S. Estimates of the fractal and Hausdorff dimensions of sets invariant under multimappings // Mathematical Notes, 1998. – Vol.63, No.2. – PP.217–224.

[213] Melnikov V.K., On the stability of center for time periodic perturbations // Trans. Moscow Math. Society, 1963. – Vol.12. – PP.1–57.

[214] Monin A.S., Polubarinova-Kochina P.Ya., Khlebnikov V.I. Cosmology, hydrodynamics, turbulence: A.A.Fridman and development of his scientific legacy, Moscow, Nauka, 1989. - 326 p.

[215] Methods of analysis of nonlinear dynamical models / S. Holodniok, A. Klich, M. Kubichek, M. Marek / Transl. from Czechish, Moscow, Mir, 1991. – 368 p.

[216] Moskalkov M.N. Analysis of dispersion properties of difference schemes for the transport equation / in *Numerical Analysis*, Kiev, Int. of Cybernetics, 1978. – PP.75–86.

[217] Moskaliuk S.S., Verlan' A.F. Mathematical Modelling of Continuous Dynamical Systems, Kiev, Naukova Dumka Publishers, 1988. 288 p. (in Russian).

[218] Moskaliuk S.S., Kuznetsov G.I., Smirnov Yu.F., Shelest V.P. Graphical Theory of Representations of Orthogonal and Unitary Groups and its Physical Applications, Kiev, Naukova Dumka Publishers, 1992. 288 p. (in Russian).

[219] Moskaliuk S.S., Gromov N.A., Smirnov Yu.F. The Trees, Contractions and Analytical Continuations Methods in Representations Theory of Classical Groups and Quantum Algebras, Kiev, Naukova Dumka Publishers, 1993. 310 p. (in Russian).

[220] Moskaliuk S.S. From Cayley-Klein Groups to Categories, Kyiv: TIMPANI Publishers, 2006, 352 p.

[221] Moskaliuk S.S. Category of Trees in Representation Theory of Cayley-Klein Groups, Kyiv, TIMPANI Publishers, 2007. 328 p.

[222] Mukhin S.I., Popov S.B., Popov Yu. P. On difference schemes with artificial dispersion // USSR Computational Mathematics and Mathematical Physics, Springer Translation, 1983. – Vol.23, No. 6. – PP.45–53.

[223] Nakagawa T. Blowing up of a finite difference solution to $u_t = u_{xx} + u^2$ // Applied mathematics & optimization - 1975-1976. – Vol.2, No.4. – PP.337–350.

[224] Nakagawa T., Ushijima T. Finite element analysis of the semilinear heat equation of blow-up type / in *Topics in numerical analisys*, 1977. - Vol.3.

[225] Nakoryakov V.E., Borisov A.A. Propagation of perturbations in media with relaxation or chamical reactions // Combustion, Explosion and Shock Waves, 1976. – Vol.12, No.3. – PP.414–422.

[226] Nakoryakov V.E, Pokusajev B.R. and Schreiber I.D. Evolution of waves in gaseous-liquid mixtures, Inst. for Thermal Physics, Novosibirsk, 1983.

[227] Nakoryakov V.E., Pokusajev B.G. and I.R. Schreiber Volnovaya Dinamika Gazo- i Parozhidkostnykh Sred // Wave Dynamics of Gas- and Vapour-Liquid Media, Energoizdat Publ., Moscow, 1990.

[228] Nesterenko V.F. Propagation of nonlinear compression pulses in granular media // Journal of Applied Mechanics and Technical Physics (PMTF), 1983. – Vol. 24. – PP.136–148.

[229] Nesterenko V.F. Impulse comression of heterogenious materials, Moscow, Nauka, 1992. – 200p.

[230] Nesterenko V.F. Dynamics of heterogeneous materials, Springer, New York, 2001. – 227 p.

[231] Nikitin A.G., Barannyk T. Solitary Waves and Other Solutions for Nonlinear Heat Equations, arXiv:math-ph/0303004 (2003).

[232] Nikolajevskiy V.N. Viscoelastisity with internal oscilations as a possible model of saismic media // Doclady AS USSR, 1985. – Vol.283, No.6. – PP.1321–1324.

[233] Nikolajevskiy V.N. On the studies of waves in seismic media // in *Problems of nonlinear seismics*, Moscow, Nauka, 1987. – PP. 190–202.

[234] Nikolis H., Prigogine J. Self-organization in nonequilibrium systems, Moscow, Mir, 1979. – 512 p.

[235] Newell A.O., Rand D.A., Russell D. Turbulent dissipation rates and the random occurrence of coherent events // Physics Letters, Ser. A., 1988. – Vol.132, No. 2,3. – PP.112–123.

[236] Novikov E.A. Dynamics and statics of the system vortex // Journal of Experimental and Theoretical Physics (ZhETF),1975. – Vol.68, No.5. – PP.1868–1883.

[237] Novikov A.N. Estimation spatially distributed dynamical processes being described by variational inequalities // Avtomatika, 1993. – No.2. – PP.45–55.

[238] Oleinik O. A. Discontinuous solutions of non-linear differential equations // Russian Mathematical Surveys (UMN), 1957. – Vol.12, No.3(75). – PP.3–73.

[239] Olver P.J. Applications of Lie groups to differential equations, Springer, New York, 1993.

[240] Olver P.J., Rosenau P. The construction of special solutions to partial differential equations // Phys. Lett. A, 1986. – Vol.114., No.3. – PP.107–112.

[241] Olver P.J. and Rosenau P. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support // Phys. Rev. E, 1995. – Vol.53, No.2. – PP.1900–1906.

[242] Olver P.J., Vorob'jev E. Nonclassical and conditional symmetries / in *CRC Handbook of Lie group analysis of differential equations*, Vol. III, ed. by Nail Ibragimov. – CRC Press Inc, 2000.

[243] Oskolkov A.P. Initial-boundary problems for the Kelvin-Poyght and Oldroytt fluid motion equation // Trudy Math. Inst. AN BSSR, 1987. – Vol.179. – PP.126–164.

[244] Oskolkov A.P. Initial-boundary value problems for equations of motion of Kelvin–Voight fluids and Oldroyd fluids // Boundary value problems of mathematical physics. Part 13, Work collection, Trudy Mat. Inst. Steklov., 179, 1988. – PP.126-–164.

[245] Ovsiannikov L.V. Group analysis of differential equations, New York, Academic Press, 1982.

[246] Ovsyannikov I.M., Shilnikov L.P. Systems with homoclinic curve of the saddle-focus // Soviet Math. Sbornik, 1986. – Vol.130(172). – PP.552–570.

[247] Palmov V.A. On some model of media with complex structure // Journal of Applied Mathematics and Mechanics, 1969. – Vol.4. – PP.768–773.

[248] Polyakova A.L. On the propagation of waves with finite amplitude in relaxing media // Acaustical physics, 1960. – Vol. VI, No.3. – PP.358–361.

[249] Polyakova A.L., Solujan S.I., Khokhlov R.V. On the quastion about the propagation of waves with finite amplitude in relaxing media // Acaustical physics, 1962. – Vol. VIII, No. 1. – PP.107–112.

[250] Peerlings R.H.J. Enhanced damage modelling for fracture an fatigue, Technische Universiteit, Eindhoven, 1999.

[251] Petrov A.A., Shananin A.A. A system analysis of economic. The problem of agregate description of the economic relations // Math. Modeling. The methods of description and investigations of the complicated systems, Moscow, Nauka, 1989. – PP.121–157.

[252] Picirelli R. Theory of the dynamics of simple fluids for large spatial gradients and long memory // Physical Review, 1968. – Vol.175, No.1. – PP.77–98.

[253] Pikovsky A., Rosenau P. Phase compactons // Physica D, 2006 – Vol.218. – PP.56–69.

[254] Pisarenko G.S. Vibration of mechanical systems involving imperfectly elastic materials, Kyiv, Naukova dumka, 1970. – 357p.

[255] Presutti E. Collective phenomena in stochastic particle systems // Lecture Notes in Mathematics, Springer Berlin / Heidelberg, 1987. – Vol. 1250. – PP.195–232.

[256] Pyatkov S.B., Riudyak V.Ya., Smagulov Sh. Of the propertiesof the solution of hyperbolic approximation of Navier-Stokes aquations //Chislennye metody meoh. eploshnoy sredy, Movosibirsk, 1982. – Vol.13, No.13. – PP.104–113.

[257] Rand D.A., Newell, Russell D. Turbulent Transport and random Oscurrence of coherent events // Physica D, 1988. – Vol.33, No.1-3. – PP.284–303.

[258] Richtmyer R.D., Morton K.W. Difference methods for initial - value problems, New York, Wiley and Sons, 1967.

[259] Rodionov V.N. Essay of geomechanics, Moscow, Nauchnij mir, 1996. – 64 p.

[260] Rudenko O.V., Solujan S.I. The propagation of waves with finite amplitude in ideal dissociative gase // Acoustical physics, 1970. – Vol. 16, No.1. – PP.161–163.

[261] Rudenko O.V., Solujan S.I., Khokhlov O.V. Problems of nonlinear acoustics theory // Acoustical physics, 1974. – Vol.XX, No.3. – PP.449–457.

[262] Ramasubramanian K., Sriram M.S. A comparative study of computation of Lyapunov spectra with different algorithms // Physica D, 2000. – Vol.139, No.1-2. – PP.72–86.

[263] Resibua P., De Lener M. Classical kinetic theory of fluids and gases, Moscow, Mir, 1980.

[264] Rosenau P. Random walker and the telegrapher's equation: A paradigm of a generalized hydrodynamics // Phys. Rev. E, 1993. – Vol.48, No.2. – R655–R657.

[265] Rosenau P. On solitons, compactons and Lagrange maps // Physics Letters A, 1996. – Vol.211, No.5. – PP.265–275.

[266] Rosenau P. On a model equation of traveling and stationary compactons // Phys. Lett. A, 2006. – Vol.356, No.1. – PP.44-50.

[267] Rosenau P., Hyman J. Compactons: solitons with finite wavelength // Phys. Rev. Letter, 1993. – Vol. 70, No.5. – PP.564–567.

[268] Rosenau P., Pikovsky A. Phase compactons in chains of dispersively coupled oscillators // Phys. Rev. Lett. – 2005. – Vol.94. – P.174102.

[269] Roache P.J. Computational Fluid Dynamics, Moscow, Mir, 1980.

[270] Rozhdestvenski B.L., Yanenko N.N. The systems of quasilinear hyperbolic equations and their applications to gas dynamics, Moscow, Nauka, 1978.

[271] Ruelle D. The Turbulent Fluids as a Dvnamical System / in *New Trends in Turbulence*, Springer-Verlag, 1991. – PP.123–163.

[272] Russell J.M. Amplitude propagation in slowly varying trains of shear-flow Instability waves // Journal of Fluid Mechanics, 1986. – Vol.170. – PP.21–51.

[273] Rudyak V. Ya. Statisticheskaya Teorija Dissipativnykh Processov v Gazach i Zhidkostiach / in *Statistical Theory of Dissipative Processes in Gases and Liquids*, Moscow, Nauka, 1987.

[274] Rudyak V. Ya., Yanenko N.W. Some nonlocal model of fluid mechanics // Math. Modelling, 1985. – Vol.6. – PP.401–412.

[275] Samarskii A.A. Homogeneous difference schemes for non-linear equations of parabolic type // Zhurnal vychislitelnoy matematiki i matematicheskoy fiziki, 1962. – Vol.2, No.1. – PP. 25–56.

[276] Samarskii A.A. Numerical schemes theory, Moscow, Nauka, 1977.

[277] Samarskii A.A., Galaktionov V.A., Kurdiumov S.P. et al. Blow-up regimes in quasilinear parabolic-type equations, Moscow, Nauka, 1987.

[278] Samarskii A.A., Gulin A.V. Stability of difference schemes, Moscow, Nauka, 1973. - 386 p.

[279] Samarskii A.A., Qalaktionov Y.A., Kurdyumov S.P., Michajlov A.L. The blow-up regimes in the problems for the quasilinear parabolic equations, Moscow, Nauka, 1987.

[280] Samarskii A.A., Popov Yu. P. Difference methods for solving of problems of gas dynamics, Moscow, Nauka, 1980.

[281] Santini P. M., Fokas A. S. The initial-boundary value problem for the Davey-Stewartson I equation; how to generate and grive localized coherent structures in multidimensions / in *Partially integrable evolution equations in physics*, edited by R.Conte and N.Boccera, NATO ASI Series, Series C – Vol.310, Kluwer, Dordrecht, 1990.

[282] Shapiro V.D., Shevchenko V.I. Strong plasma turbulence // Fundamentals of plasma physics, Moscow, Energoatomizdat, 1984. – Vol. 2.

[283] Sierre D. Propagation des oscillations dans les systems hyperboliques non lineares // Lect. Not. Math., 1987.– No.1270. – PP.265–285.

[284] Sharkovski A.N., Maistrenka Yu.L., Romanenko E.Yu. The difference equations and their applications, Kiev, Naukova dumka, 1986. -– 280 p.

[285] Shil'nikov L.P. On one case of the existence of a countable set of periodic movements // Sov. Math. Dokl, 1965. – Vol. 6. – PP.163–166.

[286] Shokin Yu. I., Yanenko N.N. Differential approximation method. Applications in gas dynamics, Novosibirsk, Nauka, 1985.

[287] Shkadov V.Ya. Solitary waves in a layer of viscous fluid // Izvestija AN SSSR, Mekh. Zhidk. i Gaza, 1977. – No. 1. – PP.63–66.

[288] Shreyder Yu.A., Sharov A.A. The systems and the models, Leningrad, Radio i svjaz., 1982. – 152 p.

[289] Shuster H.G. Deterministik chaos. Introduction, Weinheim, 1984.

[290] Sidorets V. N., Vladimivor V.A. // Proc. of the Second Int. Conference "Symmetry in Nonlinear Mathematical Physics", July 7-13 1997, Kyiv, Ukraine", 1997. – Vol. 2. – PP.409–417.

[291] Skurativskiy S.I. Autowave solutions of the model of media with spatial and temporal nonlocalities. Ph.D.thethis: 01.04.01. – Kyiv, 2002. – 169 p.

[292] Sadovskiy M.A. Selfsimilar of geodynamical processes // Vestnik AS USSR. – 1986. – No. 8. – PP.3–11.

[293] Slepjan L.I. The wave of deformation in rod with amortized mass // Mechanics of Solids, 1967. – No. 5. – PP.34–40.

[294] Soloukhin R.I. Detonation waves in gaseous media // Uspekhi Fizicheskich Nauk, 1963. – Vol. LXXX, No. 4. – PP.525–551.

[295] Starr P. Physics of negative viscosity phenomena, Moscow, Mir, 1972.

[296] Sternin V.Yu., Shatalov V.E. Differential equations on complex-analytical manyfields on canonical Maslovs' operators // UMN, 1988. – Vol.43, No.3. – PP.99-124.

[297] Struminskiy V.V., Velikodniy V.Yu. About the structure of the shock waves // Dokl. Akad. Nauk SSSR, 1982. – Vol.266, No. 1. – PP.28-31.

[298] Tabor M. Chaos and integrability in nonlinear dynamics: an introduction, New York, Wiley, 1989.

[299] Tabor M., Weiss J. Analytic structure of the Lorentz system // Phys. Rev. A, 1981. – Vol.24, No.4. – PP.2157–2167.

[300] Taniuti T., Wei C.C. Reductive perturbation method in nonlinear wave propagation // J.Ph.Soc. of Japan,1968. – Vol.24. – PP.941–946.

[301] Tartar L. Compensated compactness and applications to partial differential equations, Nonlinear Analysis and Mechanics: Heriot-Watt Symposium (R. J. Knops, ed.), Research Notes in Math., Pitman Press, 1979. – Vol. 39. – PP.136–212.

[302] Tartar L. Oscillation in nonlinear partial differential equations: compensated compactness and homogenization // Lect. Appl.Math., 1986. – Vol.23. – PP.243.

[303] Tartar L. H-measures and applications // International Congress of Mathematicians, Kyoto, Springer, 1990. – Vol.II. – PP.1215–1223.

[304] Temam R. Infinite-dimensional dynamical systems in mechanics and physics, Springer-Verlag, New York, 1988.

[305] Temam R. The Navie-Stokes equation. Theory and numarical analysis, Moscow, Mir, 1981. – 403 p.

[306] The numerical solving of the multidimentional problem of gas dynamics / ed. by S.K.Godunov, Moscow, Nauka, 1976. – 400 p.

[307] The long waves: scientific-technical progress and social-economic development / S.Yu.Glazjev, G.I.Micherin, P.N.Seslya and other, Novosibirsk, Nauka, 1991. – 224 p.

[308] Tichonov A.H., Samarski A,A. The equations of mathematical physics, Moscow, Nauka, 1976. – 736 p.

[309] Tsytovich B.N. The theory of plasma turbulence, Moscow, 1971. - 423 p.

[310] Tsytovich B.N. The development of concepts about plasm turbulence // Uspechi Phys. Nauk, 1972. — Vol.108, No. 1. – PP.143–176.

[311] Ulenbek J., Terd J. The lectures on the statistical mechanic. Moscow, Mir, 1965. – 307 p.

[312] Vachnenko V.A., Danilenko V.A., Kulich V.V. The elements of self-organization and nonlinear wave processes in the natural media with a structures // Preprint / Geophys. Inst. of Ukr. AS, Kiev, 1991. — 44 p.

[313] Vakhnenko V.A., Kulich V.V. Long-wave processes in periodic media // J.Appl. Mech. and Techn. Phys., 1992. – Vol.32. – PP.814–820.

[314] Vidybida A. K. Evolution operator for the Bogolyubov (BBGKY) hierarchy. Lattice systems // Theoretical and mathematical physics, 1986. – Vol.68, No. 1. – PP.69—87.

[315] Vishik M.I., Fursikov A.V. Mathematical problems of statistical hydromechanics, Moscow, Nauka, 1980. – 440p.

[316] Volosov K.A. Invariant properties of the ansatz of the Hirota method for quasilinear parabolic equations // arXiv:math-ph/0103014 v1

[317] Vladimirov V.A. // Trans. Instit. Mathematics, NAS of Ukraine, 2000. – Vol. 30(I). – PP.5–10.

[318] Vladimirov V.A. Modeling system for relaxing media. Symmetry, restrictions and attractive features of invariant solutions // in *Proceedinds of Third International Conference "Symmetry in Nonlinear Mathematical Physics"*, (12–18 July, 1999, Kyiv), Editors A.G. Nikitin and V.M. Boyko, Kyiv, Institute of Mathematics, 2000. – Vol.30, Part 1. – PP.239–257.

[319] Vladimirov V.A. On the coefficients of canonical Poincaré form corresponding to 3-dimensional dynamical system with degenerated linear part // Gerald of Kiev National University, 2000. – No. 1. – PP.66–74.

[320] Vladimirov V.A. Nonlinear wave structures in models of relaxing media: Thethis doc. ph.-m. of science: 01.04.01. – Kyiv., 2000. – 300 p.

[321] Vladimirov V.A. Periodic and soliton-like regimes in the gradient model of structured media // Dopovidi NAS of Ukraine, 2004. – No. 2. – PP.13–18.

[322] Vladimirov V.A. Compacton-like solutions of the hydrodynamic system describing relaxing media // Rep. Math. Physics, 2008. – Vol.61, No.3. – PP.380–400.

[323] Vladimirov V.A. Compacton-like solutions of the nonlocal hydrodynamic-type // 2008, arXiv:0804.2022v1 [nlin.PS]

[324] Vladimirov V.A., Kutafina E.V. Exact travelling wave solutionsof some nonlinear evolutionary equations // Rep. Math. Physics, 2004. – Vol.54, No.2 – PP.261–271.

[325] Vladimirov V., Kutafina E., Pudelko A. Construction Soliton and Kink Solutions of PDE Models in Transport and Biology // SIGMA, 2005. – Vol. 1. – Paper 16.

[326] Vladimirov V.A., Kutafina E.V. Towards an approximation of solitary-wave solutions of non-integrable evolutionary pdes via symmetry and qualitative analysis // Rep. Math. Physics, 2005. – Vol. 56, No.3. – PP. 421–436.

[327] Vladimirov V.A., Kutafina E.V. Analytical description of the coherent structures within the hyperbolic generalization of Burgers equation // Rep. Math. Physics, 2006. – Vol.58, No.3. – PP.465–476.

[328] Vladimirov V.A., Maczka Cz. Exact solutions of generalized Burgers equation, describing travelling fronts and their interactions // Rep. Math. Physics, 2007. – Vol. 60, No.2. – PP.317-328.

[329] Vladimirov V., Maczka Cz. On the localized wave patterns supported by convection-reaction-diffusion equation // Rep. Math. Physics, 2010. – Vol.65. No.2. – PP.141-156.

[330] Vladimirov V.A., Sidorets V. M. On the stochastic autowave solutions nonlinear hydrodynamic model of continuum medium taking into account relaxation effects // Dop. NAS of Ukraine, 1999. – No.2. – PP.126–131.

[331] Vladimirov V.A., Sidorets V.N., Skurativskyy S.I. Complicated travelling wave solutions of a modelling system describing media with memory and spatial nonlocality // Rep. Math. Phys, 1999. –Vol. 44, No.1-2. – PP.275–282.

[332] Vladimirov V.A., Skurativskyy S.I. On invariant wave patterns in non-local model of structured media / in *Proc. International Conf. "Symetry in Nonlinear Mathematical Physics"*, Vol.III, Kiyv (Ukraine), 1999. – PP.239–244.

[333] Vladimirov V.A., Skurativskii S.I. On invariant wave patterns in nonlocal model of structured media / in *Proceedinds of Third International Conference "Symmetry in Nonlinear Mathematical Physics"* (12–18 July, 1999, Kyiv), Editors A.G. Nikitin and V.M. Boyko, Kyiv, Institute of Mathematics, 2000, V.30, Part 1. – PP.231–238.

[334] Vladimirov V.A., Skurativsky S.I. Soliton-like solutions and other wave patterns in the nonlocal model of structured media // Rep. Math. Phys., 2000. – Vol.46, No.1-2. – PP.287–294.

[335] Vladimirov V.A., Skurativskiy S.I. Autowaves and soliton-like solutions of the nonlocal models of structured media // Dop. NAS of Ukraine, 2000. – No.11. – PP.7–12.

[336] Vladimirov V.A., Skurativskiy S.I. Investigations of the autowave solutions of a model of structured media with spatial and temporal nonlocality // Rep.Math.Phys., 2002. – Vol.49.– PP.405-414.

[337] Vladimirov V.A. Compacton-like solutions of the hydrodynamic system describing relaxing media // Rep. Math. Phys., 2008. – Vol.61, No.3. – PP.381-400.

[338] Vladimirov V.S. The equations of mathematical physics, Moscow, Nauka, 1976. – 528 p.

[339] Weyl H. Symmetry, Princenton university press, 1952.

[340] Whitham G. Linear and nonlinear waves, Wiley, New York, 1973.

[341] Wilkinson Y.L. Non-newtonian fluids, Moscow, Mir, 1964.

[342] Wilson A.J. Entropy methods of modeling complex systems, Moscow, Nauka, 1978.

[343] Witten E. Physics and Geometria // Proc. of the Intern. Congress of Mathematics, 1986, Berkley, USA-AIM, 1987. – PP.267–303.

[344] Wolf K., Morris H.C. Chaotic solutions of systems of the first order partial differential equations // SIAM J.Math.Anal., 1987. – Vol.18, No.4. – PP.1040–1063.

[345] Wong A.Y., Cheng P.Y. Three dimensional self-collapse of Lingmuir-waves // Phys.Rev.Lett.,1984. – Vol.52, No.14. – PP.1222–1225.

[346] Yanenko N.N., Sidorov A.F., Shapeev V.P. The method of differential constraints and its application in gas dynamics, Novosibirsk, Nauka, 1984.

[347] Yasnikov G.P, Galperin L.G., Kutiavin Z.I. // Inzh.-Fiz. J., 1980. – Vol.39. – P.334.

[348] Young L.C. Lectures on the calculus of variations and optimal control theory, W.B.Saunders, Philadelphia, 1980. – 337 p.

[349] Zabrodsky H., Peleg S. Avnir D. Continuous symmetry measure. // J. American Chem. Soc., 1992. – Vol. 114. – PP.7843–7851.

[350] Zacharov V.E. The wave collapses in the continuous medium physics // The problems of physical kinetics and solid body physics, Kiev, Naukova dumka, 1990. – 488 p.

[351] Zaharov V.E., Kuznetsov E.A. Quasiclassical theory of the three-dimensional wave collapses //GETF, 1986. – Vol.91, No.4. – PP.1310.

[352] Zaharov V.E., Pushkarev A.N.., Rubenchik A.M. Kinetics three-dimensional Lingmuir collapses//GETF, 1989. – Vol. 96, No.2(8). – PP.591–603.

[353] Zaharov V.E. Wave collapses in physics of continuous media // Problems of physical kinetics and physics of solid states, 1990. – PP.469–486.

[354] Zajtsev V.F., Polyanin A.D. A reference book of the ordinary differential equations, Moscow, Fiz.-Mat. Lit. Publ., 2001.

[355] Zeldovich Ja.B., Barenblatt G.I., Librovich V.B., Machviladze G.M. The mathematical theory of combustion and explosions, Moscow, Nauka, 1980. – 478 p.

[356] Zgurovsky M., Melnik V. Nonlinear analysis and control of physical process and fields, Berlin, Springer-Verlag, 2004.

[357] Zhang F., Gronig H. Spin detonation in reactive particles-oxidizing gas flow // Phys. of Fluids A: Fluid Dynamics, 1991. – Vol. 3, No. 8. – PP.1983–1990.

[358] Zhigulov R.N. To the theory of dissipation processes in the turbulent flow of incompressible fluid // Turbulent flows. - Nauka, Moscow, 1974. – PP.75–84.

[359] Zubarev D., Tishchenko S. Nonlocal hydrodynamics with memory // Physica, 1972. – Vol.59, No.2. – PP.285–304.

[360] Zubarev D.N. Nonequlibium statistical thermodynamics, Moscow, Nauka, 1975. – 320 p.

[361] Zubarev D.N. Statistical thermodynammics of turbulent transport processes // Theoretical and mathematical physics, 1982. – Vol.53, No.1. – PP.93–107.

[362] Zubarev D. N., Khazanov A. M. Generalized Fokker–Planck equation and construction of projection operators for different methods of reduced description of nonequilibrium states // Theoretical and mathematical physics, 1978. – Vol. 34, No.1. – PP.69–80.

[363] Zubarev D.N., Tokarchuk M.V. Statistical hydrodynamics of the condensed media with ionic systems // Thermodynamics of irreversible processes, Moscow, Nauka, 1992.

[364] Zwanzig R. Ensemble method in the theory of irreversibility // J.Chem.Phys., 1960. – Vol.33. – PP.1338–1341.